THE UNIVERSITY OF ALBERTA

INDIVIDUALIZED INSTRUCTION IN GRADE SEVEN MATHEMATICS: PUPIL ACHIEVEMENT AND GROUPING PROCEDURES

BY

AUDREY KATHLEEN SUNDE

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE

DEGREE OF MASTER OF EDUCATION

DEPARTMENT OF SECONDARY EDUCATION

EDMONTON, ALBERTA

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Individualized Instruction in Grade Seven Mathematics: Pupil Achievement and Grouping Procedures" submitted by Audrey Kathleen Sunde in partial fulfilment of the requirements for the degree of Master of Education.

ABSTRACT

The study was a part of the Hardisty Project which developed an individualized mode of instruction for grade seven mathematics. The purpose of the study was twofold. The first purpose was to determine the effectiveness of the individualized mode of instruction with respect to pupil achievement within the cognitive domain. The second purpose was to assess the grouping procedures used in the individualized mode of instruction. These grouping procedures enabled each student to receive instruction in one of three groups, Basic, Intermediate or Advanced, for each topic studied. Each of three topics, rational numbers and fractions, operations with rational numbers, and decimals, was divided into two Phases. During Phase I all students studied Intermediate level objectives concerning the topic. The student's achievement of these Intermediate objectives determined which of the three groups the student entered for Phase I. Depending on which group the student entered, he would study Basic, Intermediate, or Advanced level objectives.

The achievement of the students receiving the individualized (experimental) mode of instruction was compared to that of students in a control group which received regular classroom instruction. It was found that there was no significant difference between the achievement of students in the experimental group when achievement was measured by a standardized mathematics test. However, the students in the experimental group had significantly better achievement scores when achievement was measured by a mathematics achievement

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The cooperation received from the other members of the Hardisty Project, B. G. te Kampe and M. Westrom, and from all the teachers and students involved in the Project is gratefully acknowledged.

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CHAPTER I

THE PROBLEM

I. INTRODUCTION

A historical review of individualized instruction indicates that:

> Formal learning began very much as an individual affair--that is, pupils came to school to receive instruction individually from the teacher. Education was generally for a select few, therefore, fewer pupils attended school. This made possible the provision of individualized instruction for these students. For example, in the one-room school, pupils proceeded on an individual basis rather than as intact groups. As educational advantages were offered to a larger proportion of the population, it became necessary to deal with pupils in grade-level groups, and individualized instruction diminished. Since then, however, as awareness of individual differences among pupils increased, many efforts were made to individualize instruction even within the context of schools offering mass education (Scanlon and Bolvin, 1969, p. 2).

Recognizing the necessity of meeting the needs of individual students, the grade seven mathematics teachers at the Hardisty Junior High School, Edmonton, Alberta, decided to implement a mode of instruction which would cater to individual differences among students. The so called 'Hardisty Project' had as its central task the development of an individualized mode of instruction. Within this Project, the present study examined the effect of this instructional mode upon the mathematical achievement of the Hardisty Junior High School pupils.

II. BACKGROUND OF THE PROBLEM

Evaluation procedures to determine the effectiveness of educational programs are an essential aspect of the development of these programs. A program developed by the Learning Research and Development Center of the University of Pittsburgh, and Research for Better Schools Inc., a Regional Educational Laboratory based in Philadelphia, known as Individually Prescribed Instruction (IPI) has been evaluated by answering the following questions. How well do the materials do the job of helping the pupils learn? How adaptable are they to the varying requirements of each pupil? How does the individual respond to this type of approach? How does the teachers' role change? What happens to the pattern of staffing the school? What are the effects on the community and what are its expectations of the system? What are the variations in cost? (Scanlon and Bolvin, 1969, p. 7).

The evaluation of the Hardisty Project encompassing three different studies involved the investigation of some of the questions mentioned above. The present study was concerned with determining the effectiveness of the individualized mode of instruction with respect to pupil achievement. The use of grouping procedures used in this mode of instruction as they affect the students mathematical achievement was also examined. A comparison thesis written by B. G. te Kampe is concerned with describing

the role of the teacher using this mode, and comparing it to the role of the teacher using regular classroom instruction. A second comparison thesis was written by M. Westrom. The thesis gives a description of the rationale for the many ideas implemented in the development of this mode of instruction, and determines the feasibility of implementing such an instructional program.

III. STATEMENT OF THE PROBLEM

The purpose of the study was twofold. First was to determine the effectiveness of the individualized mode of instruction with respect to pupil achievement in the cognitive domain. The effects of reading ability and I.Q. level on pupil achievement were considered. The second purpose was to assess the grouping procedures used for instructional purposes implemented in the individualized mode of instruction. The group membership of each student was determined by their achievement of behaviorially stated objectives. Objectives studied in the different groups varied with respect to difficulty and necessity of achievement for further advancement throughout the topic. Group membership was flexible in that it could be changed for each topic covered.

IV. SIGNIFICANCE OF THE PROBLEM

The importance of meeting individual differences among students is recognized in the following statement.

> One of the most pressing needs in education today is the adaption of instruction to individual characteristics and backgrounds (Bolvin and Glaser, 1968, p. 828).

Modes of instruction which provide for individual differences are being formulated and studied both locally and internationally. An evaluation of these modes of instruction is considered a major part of all of these projects. The significance of evaluation of new curricula is discussed by Stake.

> One of the largest investments being made in U.S. education today is in the development of new programs. School officials cannot yet revise a curriculum on rational grounds, and the needed evaluation is not underway. What is to be gained from the enormous effort of the innovators of the 1960's if in the 1970's there are no evaluation records? Both the new innovator and the new teacher need to know. Folklore is not a sufficient repository. In our data banks we should document the causes and effects, the congruence of extent and accomplishment, and the panorama of judgements of those concerned (1968, p. 336).

Evaluation as it is used above refers to a full or complete evaluation of the new curriculum or program. Within this full evaluation, several smaller studies must be made which contribute significantly to the full evaluatión. The problem investigated in this study was not a complete evaluation of the whole Project. It dealt

with only two aspects of evaluation: the effectiveness of the individualized mode of instruction with respect to pupil achievement within the cognitive domain, and the effectiveness of the grouping procedures, based on pupil achievement, implemented in this mode of instruction. Assessment of pupil achievement is one of the most significant aspects of an evaluation of an instructional program.

V. DEFINITIONS

Individualized Instruction -- a style of teaching which allows students to work at their own level of ability and rate of learning, either independently or in small groups as the situation demands. It is the style of teaching which was used with the experimental group.

Regular Classroom Instruction -- a style of teaching which centers instruction around the 'average student' in a group situation. It is the style of teaching which was used with the control group.

Behaviorial Objective -- an objective which states the behavior required of the learner which is accepted as evidence that the learner has achieved the objective; the conditions under which the behavior is expected to occur, and the criterion for achievement of the objective.

Achievement Tests -- the tests constructed by the writer to test achievement of the specific subject matter taught during the experiment.

Standardized Tests -- The Cooperative Mathematics Test - Arithmetic Form A or B used as pre- and post-tests respectively to test general mathematical concepts.

VI. EXPERIMENTAL SETTING

The individualized mode of instruction developed was presented to nine grade seven mathematics classes at the Hardisty Junior High School, Edmonton, which formed the experimental group. A comparison study was made to compare the mathematics achievement of the students receiving the individualized mode of instruction with that of students receiving regular classroom instruction. Seven grade seven mathematics classes at another junior high school in the city formed the control group for the comparison study and received regular classroom instruction.

VII. LIMITATIONS

(1) Neither the experimental nor the control group were random samples of grade seven students. Thus, the results of this study are not statistically inferrable to any larger existing population.

(2) Neither the teachers nor the treatment were assigned randomly to the classes. These were determined before the beginning of the experiment.

(3) The results of the study depended on the extent to which the teachers consistently taught in either mode

of instruction as used in the experimental and control situations. Reference should be made to the thesis written by B. G. te Kampe with respect to this limitation. The role of the teacher using both modes of instruction is described and compared.

(4) The study was concerned only with grade seven students, and the subject matter was limited to rational numbers, rates and percent.

(5) Mathematics was the only subject in which the students used an individualized mode of instruction. Adjustment to this mode of instruction was thus required of the experimental group each time they had a mathematics class.

VIII. OUTLINE OF THE REPORT

The present chapter is an introduction to the study. A review of the related literature is given in Chapter II. Chapter III consists of a detailed description of the experimental design of the study and the development of the testing instrument.

Chapter IV consists of the statistical analysis of the results. A summary of the investigation along with the conclusions and implications for further research are presented in Chapter V.

CHAPTER II

THE RELATED LITERATURE

I. INTRODUCTION

The review of the related literature is divided into three sections. The first section is concerned with the theory of curriculum evaluation. The second section consists of a review of some current projects concerning individualized instruction. In this review, an emphasis is placed on the pupil-achievement and grouping-procedure aspects of these projects, as these aspects of the Hardisty Project were the major concern of the present study. The third section contains a discussion on criteria for the selection and construction of testing instruments.

II. EVALUATION

What is educational evaluation? It is broadly defined by Cronbach as "the collection and use of information to make decisions about an educational program" (1963, p. 672). Many types of decisions must be made, and many varieties of information are useful.

Grobman says that:

> If the evaluation is to be useful, each project must develop its own unique pattern, reflecting the interests and circumstances of the project and the clientele for whom the curriculum is patterned (1968, p. 13).

CHAPTER II

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I. INTRODUCTION

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II. EVALUATION

What is educational evaluation? [illegible] information to make decisions about an educational program [illegible] types of decisions must be made and [illegible]

[illegible]

The purpose of educational evaluation is referred to by Stake as being expository:

> . . . to acquaint the audience with the workings of certain educators and their learners. It differs from educational research in its orientation to a specific program rather than variables common to many programs. A full evaluation results in a story, supported perhaps by statistics and profiles. It tells what happened. It reveals perceptions and judgements that different groups and individuals hold--obtained, I hope, by objective means. It tells of merit and shortcoming as a bonus, it may offer generalizations ('The moral of the story is . . .') for the guidance of subsequent educational programs (1967, p. 5).

The roles which educational evaluation has to play are described by Scriven (1967).

Formative and Summative Evaluation

Scriven explains evaluation as having two roles to play. He refers to these roles as 'Formative Evaluation' and 'Summative Evaluation'. Formative evaluation is an on-going improvement of the curriculum while the curriculum is still being developed. It results in changes being made to the curriculum. Summative evaluation is a final evaluation of the educational instrument or program which permits several conclusions and decisions to be made. It may serve to enable administrators to decide whether the entire finished curriculum, refined by the formative evaluation, represents a significant advance on the available alternatives to justify the adoption of this new curriculum by a school system.

Cronbach also discusses the various roles of evaluation. He states that,

> Evaluation, used to improve the course while it is still fluid, contributes more to improvement of education than evaluation used to appraise a product already placed in the market (1963, p. 675).

Scriven does not feel that one can accept this statement asserting the greater importance of formative evaluation as compared to summative evaluation. He feels that curriculum projects must attempt to make the best use of evaluation in both roles.

Intrinsic and Pay-Off Evaluation

Scriven (1967) also discusses two procedures which may be used in evaluation. They are 'intrinsic evaluation' and 'pay-off evaluation'. Intrinsic evaluation is an evaluation of the educational instrument or program itself. This would involve evaluation of the content, goals, grading procedures, teacher attitude, grouping procedures, etc. Pay-off evaluation is an evaluation of the outcomes or the effects of the educational instrument or program on the pupil. It would involve an appraisal of the differences between pre- and post-tests, or between experimental group tests and control group tests. Intrinsic and pay-off evaluation may be either formative or summative, as they are procedures to use rather than roles of evaluation.

Comparison evaluation procedures which may be used

in pay-off evaluation were discussed by Scriven (1967) and Cronbach (1963).

Comparison and Non-Comparison Evaluation

Scriven and Cronbach do not agree on the values of comparative evaluation (comparing an experimental group to a control group). Cronbach feels there are definite weaknesses of group comparisons and that non-comparative evaluation is better. Non-comparative evaluation means to determine the post-course performance of a well described group with respect to many important objectives which were determined before the treatment began.

One weakness which Cronbach feels results from comparison studies is that we do not obtain data to determine where a new program is good and where it is not as good as an alternative. Scriven admits this, but explains that non-comparative evaluation is not the answer. What is needed are more control groups so that more short-run studies may be made. These studies would determine the advantages and the inadequacies of the new program. A single comparison study provides only a summative evaluation of a whole program and Scriven admits this is not enough. He indicates this in the following statement:

> . . . the impact of his (Cronbach's) article is to suggest the unimportance of the control group study, whereas the case can only be made for its inadequacy as a total approach to the whole of curriculum research (1967, p. 67).

Another weakness of comparative evaluation which Cronbach discussed is the impossibility to control for all variables between the two groups. Students know if they are in an experimental group and it is quite impossible to neutralize the biases or enthusiasm of the teacher. Scriven, feels however, that we can design a study to control for these variables by using more than one comparison group. He states that,

> If we use only one control group, we cannot tell whether it's the enthusiasm or the experimental technique that explains the difference. But if we use several experimental groups, we can estimate the size of the enthusiasm effect (1967, p. 68).

Implications to the Present Study

While formative evaluations were performed during the development of the Hardisty Project, the present study was concerned with a summative evaluation of the program developed. Both intrinsic and pay-off evaluation procedures were used in the summative evaluation. Intrinsic evaluation was involved in that a treatment variable within the individualized mode of instruction was examined. The treatment variable examined was the grouping procedure used in this mode of instruction. Pay-off evaluation was involved in that the effect of the individualized mode of instruction on pupil achievement was assessed. Both pre- and post-tests, and experimental and control group achievement test results were compared. A comparison study of the

pupil achievement of the experimental and control groups was made, for as Scriven said,

> When we come to evaluate the curriculum, as opposed to merely describing its performance, then we inevitably confront the question of its superiority or inferiority to the competition (1967, p. 64).

To form final judgements on an instructional program, it is essential to have compared it to alternative types of instruction.

III. EXPERIMENTAL PROJECTS

Several individualized instruction projects now in progress have been discussed by Edling (1970). He suggests that they all have one thing in common:

> The traditional form of instruction, during which all students are taught the same skill or concept at the same time, is being abandoned. Instead, cooperating professionals (administrators, teachers, and support personnel) are creating environments in which individuals proceed at a self-determined pace, often in self-selected subjects, to achieve self-evaluated and self-satisfying goals.
>
> The effects of these efforts are impressive, not necessarily in terms of performance on standardized tests (although gains here are sometimes rather startling) but rather in terms of change in the behavior of learners and teachers. There is an almost unanimous report of renewed interest in school and educational activities. Traditional disciplinary problems have virtually disappeared. There is a major reduction in truancy and drop-out, and attendance is improved, i.e. less "illness", "travel", and other reasons for absentieism. An increasing number of high scores are appearing on specific achievement measures. Teachers report working harder than ever before, but they are more satisfied because they feel they are doing more for their students (1970, pp. 13-14).

Five projects concerning individualized instruction in mathematics are reviewed in this section. In this review, an emphasis is placed on the pupil-achievement and grouping-procedure aspects of these projects, as these aspects of the Hardisty Project were the major concern of the present study.

Individually Prescribed Instruction (IPI). The Individually Prescribed Instruction (IPI) Project of the Learning Research and Development Center, University of Pittsburgh, represents an investigation into the requirements for and the problems encountered in developing a system for individualizing instruction at the elementary school level. Individualization, as conceived by the IPI project, is defined by Bolvin and Glaser as "the adaptation of the educational environment to individual differences; put another way, the use of information about individual differences to prescribe appropriate educational environments" (1968, p. 829).

IPI works toward managing instruction so that every pupil's work can be evaluated daily, and guided by a teacher written prescription; a plan for a student to improve or master a particular skill or objective is determined. There are five building blocks which enable the teacher to effectively diagnose skills and monitor pupil progress. They are listed by Devoky as:

(1) Behaviorial Objectives,

(2) Placement Tests,

(3) Pre-Tests,

(4) Post-Tests,

(5) Curriculum Embedded Tests (1969, p. 44).

Behaviorially stated objectives are presented in a sequential order for all content to be covered. Within each sequence, the objectives are grouped in sub-sequences to provide break points for further advancement or to go on to another area.

The rate of speed at which a pupil progresses through the material depends upon his own capacities. He is placed on the continuum (the sequential order of material presented) by taking both the placement test and the pre-test. The assignments given to each student are indicated by a prescription to fit his individual needs, as diagnosed by the teacher from the results of the pre-test. This prescription is an individual lesson plan for each student each day. The students mastery of the objectives is determined by the curriculum embedded tests and the post-tests. The curriculum embedded tests are short tests which provide specific data on the mastery of each specific objective. The post-tests are parallel forms of the pre-tests. The pupil is required to perform at the 85 percent level. The students spend most of their time working independently.

There is no homogeneous grouping for classes using IPI materials. Pupils are heterogeneously grouped in a self-contained classroom. If the need arises, pupils from the same or different classrooms are grouped for instructional purposes for a short period of time. These pupils have similar problems relating to a common skill or objective.

Devoky summarized what IPI is by stating:

> IPI does not so drastically change _what_ is done in schools, as _how_ it is done. Young IPI students don't learn a different curriculum from other youngsters; they don't necessarily score higher on standardized tests. Rather they are introduced to an approach to learning where the learner is the focus, where a student moves at his own pace in his own style. The IPI teacher ceases to be a performer and group disciplinarian, and becomes instead, a diagnostician, prescriber and guide to the special learning progress of the individual. And because students are loosed from the pace of the class, they begin to spread out, according to learning rates (1969, p. 44).

Evaluation procedures are an important aspect of the IPI Project. To answer the question: 'How does this innovation work in the pilot schools?', five kinds of data are collected. Included in these are information about the materials, the pupils, the teaching staff, the community setting, and the Individually Prescribed Instruction concept as an educational system.

Aspects of IPI which are more relevant to the present study were discussed by Lipson (1967). He described the results of IPI in terms of pupil achievement.

The standardized test results of pupils using IPI materials at Oakleaf school were given. Oakleaf is one of the main experimental schools using IPI materials. First and second grade students were almost all over the eightieth percentile. The third and fourth grades were average, while the fifth and sixth grades had many students below the fortieth percentile. The low percentile rating was explained by stating that many of the upper grade students were not exposed to all of the usual sixth grade curriculum, as they were making up deficiencies in their mastery level of lower grade objectives. Thus, although they did not do so well on standardized tests, they were increasing their understanding of earlier work. When the IPI sixth grade students were in grade seven, there were no differences noted between them and other students. It was found that summer retention was very high in the IPI program. This was attributed to the mastery criterion for progression.

Lipson gave a result found by Yeager and Lindvall (1966), that there is almost a complete lack of correlation of rate of progression in the program with I.Q. I.Q. was mainly related to the ability to leap over deficiencies in the instructional process.

The following table compares the pupil achievement of an IPI school with a non-IPI (control) school on arithmetic subtests. The original table as given by Glaser (1969) also included a comparison of reading scores.

TABLE I

COMPARISON OF AN IPI SCHOOL TO A NON-IPI (CONTROL) SCHOOL FOR ARITHMETIC SUBTESTS

Gr.	No. of Pupils		Arithmetic Concepts		Arithmetic Problem Solving	
	IPI	Non-IPI	IPI	Non-IPI	IPI	Non-IPI
IV	62	86	35.4	34.2	33.1	32.6
V	53	85	41.0	40.6	41.8	40.7
VI	59	85	57.6	45.2	50.2	46.4

(1969, Table 3).

The results shown in the table indicate that although there was a small difference between IPI and non-IPI school results, the IPI school was consistently ahead in both sub-tests at all three grade levels.

Individualized Mathematics Instruction (IMU). The National Board of Education of Sweden started a project on Individualized Mathematics Instruction (IMU) in the fall of 1963. The entire project will not be completed until 1971. The goals of the project, as outlined by Oreberg, are to:

> (a) construct and test material for self-instruction in mathematics,
> (b) develop suitable instructional methods to use with this material,
> (c) study how pupils should be grouped and teachers employed to attain the maximum effect when using this material,

(d) measure, with the help of the material constructed, the effects of completely individualized instruction (possibly by comparison with conventional class instruction) (1968, p. 1).

The self-instructional materials allows the students to work at their own pace. Nine modules of material were constructed, covering the curriculum for the upper department of the comprehensive school. Subject matter within each module was structured in terms of degree of difficulty.

Each module contains three components of which the two last ones consist of three different levels of material based on degree of difficulty. All students work through the first component (A) at their own pace. When component A is completed a diagnostic test is written. On the basis of the information obtained from this test, a student is placed in one of the three levels of the second component (B) which best suits his needs. The content material of these three levels of component B are of a general character, but vary in difficulty. At the completion of component B, the students write a diagnostic-prognostic test. This test determines which level of component (C) the students will work in. A prognostic test is written at the completion of component C. A fourth component (D), which does not belong to the basic course, contains material for repetition in connection with the diagnostic tests, individual work (independent tasks), group work, and

group discussions. As each module requires approximately one month in time, the grouping procedures occur about ten times a year. This reduces the risk of assigning a student once and for all to a particular level.

Formative evaluation of the materials constructed occurs regularly. Pilot studies are conducted and revision of the materials results. Summative evaluation will also occur. An investigation into the effects of the IMU will include comparisons between six different organizational models of IMU. These six models are variations of the number of teachers and teacher's assistants participating in an instructional unit. Also to be compared are the IMU group and a control group selected at random from all classes in Sweden where instruction is traditional and the IMU system is not applied.

A full investigation of the project is planned to be completed during 1970-71. Included in this investigation will be a report planned to include the pupils knowledge and proficiency in mathematics (based on an arithmetic test, standardized test, and a final test based on the definition of the aim of the project) along with experiments concerning the flexible grouping and teaching teams. Also to be investigated are the effects of the project on student attitude and ability to accept responsibility, the effect on the teachers and teacher's assistants, the problems encountered in drawing up the curriculum, the

cost of implementation, and other effects to ascertain whether the IMU material gives better prospects of individualizing instruction than the traditional method of one teacher--one class.

Bloom. A strategy for mastery learning was developed by Bloom. He states that

> If the students are normally distributed with respect to aptitude, but the kind and quality of instruction and the amount of time available for learning are made appropriate to the characteristics and needs of each student, the majority of students may be expected to achieve mastery of the subject (1968, p. 3).

The majority of the students who can achieve mastery are the top ninety-five percent of the normal aptitude distribution. Bloom says the top five percent of the normal aptitude distribution have a special talent for learning the subject. The bottom five percent may have a special disability for a particular learning. The ninety percent of the students left can achieve mastery of a subject provided they are allowed enough time and the appropriate type of help.

The needs of the individual students must be met. Various types of instructional materials (textbooks, workbooks, programmed instruction, audiovisual methods, and academic games, etc.) may serve as a means of helping individual students at selected points in their learning process.

The strategy for mastery learning as developed by Bloom and his associates (1968) included the specification of behavioral objectives and content of instruction. The objectives indicated the criterion for mastery of that objective. Appropriate instructional techniques must then have been used to ensure mastery of the objectives.

In the study conducted by Bloom, the teacher taught the course in the same way as before with respect to content, time, materials and methods. This regular instruction was supplemented by formative evaluation. The formative evaluation was a brief diagnostic-progress test given at the completion of a learning unit to determine whether or not a student had mastered the unit. If he had not mastered the unit, the test would determine what he must do to master it, as particular points of difficulty would be revealed by this test. For those who did achieve well, the formative evaluation would reinforce their learning and assure the student his present mode of learning was appropriate. It was found that students responded best to the diagnostic results when they were referred to particular instructional materials or processes intended to help them correct their difficulties.

An investigation of the effects of the strategy for mastery learning was quite limited, but the results were very encouraging as evidence of its effectiveness

with respect to mastery of learning was found. On a mathematic achievement test written in 1965, before the strategy was used, approximately twenty percent of the students received a grade of A on the test. In 1966, after the strategy was used, eighty percent of the students achieved an A on a parallel examination. Final results of a 1967 summative evaluation instrument which was parallel to the tests of 1965 and 1966, showed that ninety percent of the students achieved a grade of A.

These results indicated that the strategy for mastery learning helped more students attain mastery (a grade of A). It should be noted however, that no indication of the similarity of the groups compared was given.

Affective consequences of the strategy were also noted. Students did well on the subject and began to like it. They were motivated for further learning and their self-concept was positive. They were receiving positive recognition for their achievements.

Mortlock. Mortlock (1969) made a study on provision for individual differences in eleventh grade mathematics using flexible grouping based on achievement of behaviorial objectives. A classroom management plan was formed to provide for the differences between individual students.

Behaviorial objectives were written for all content

to be covered in each topic at three difficulty levels. Basic objectives included the minimal behavior required to progress through a topic. Intermediate objectives included more difficult behavior and more ideas, and were expected of the average student. Advanced objectives included still more difficult behavior and more ideas; and were expected to be achieved by the superior student.

Each topic was taught in two phases. During Phase I, the students received regular classroom instruction from the teacher. Achievement was sought at the Intermediate level. At the completion of Phase I, a post-test was written on the Intermediate objectives. Depending on the student's achievement on this test, he was placed in one of three groups for Phase II of the instruction.

The levels of objectives were the basis for differentiation of what was expected of and provided for in the three groups. The groups were designated as the basic group, the intermediate group, and the advanced group. During Phase II, the students received small group and individual instruction on the objectives corresponding to their group. At the completion of Phase II, post-tests relevant to each group were given. Regrouping was permitted at the end of each topic, as the achievement in Phase I indicated to which group the student would belong in Phase II.

Mortlock commented that,

> This approach appeared to present the following advantages:
> (1) It provided for whole class activity which could help maintain class unity.
> (2) It gave all students the opportunity to demonstrate differential achievement on different topics and to be grouped accordingly.
> (3) It provided students later entering the basic group with exposure to content at the intermediate level without finally expecting them to achieve at this level.
> (4) It provided work of reasonable challenge for more able students which may not have been the case if the initial teaching had been aimed solely at achievement of basic objectives (1969, pp. 60-61).

The evaluation of the Mortlock study included the answering of two questions which are relevant to the present study. They are:

> (1) What was the relevant effectiveness with respect to mathematical achievement of the form of classroom management and the behaviorial objectives when compared with the conventional approach to mathematics instruction?
>
> (2) Was there a need for flexible grouping, or could students have been pre-grouped on the basis of some initial criteria? (1969, pp. 94-95).

The classes taught the previous year by Mortlock were used as a control group with respect to achievement. It was found that there was no significant difference between the mathematics achievement of the experimental and control groups as measured by the teacher-constructed eleventh grade mathematics final examination.

Achievement on subtests of this final examination were also compared at the three difficulty levels. There was no significant difference between the experimental and

control groups with respect to the basic and intermediate objectives. The control group did significantly better on the advanced objectives subtest. This was explained by the fact that all students in the control group had been exposed to the material relating to the advanced objectives, while only the advanced group of the experimental group were exposed to these objectives.

There was no statistically significant difference between the percentile ranks of the mean algebra achievement of the experimental students at the beginning and at the end of the eleventh grade as measured by the Cooperative Mathematics Tests - Algebra I and Algebra II respectively.

Mortlock concluded that:

> The overall conclusion to be drawn from the above findings is that the classroom use of behaviorial objectives and the setting of levels of expectations for students which were appropriate to their demonstrated achievement (grouping for each topic on the basis of achievement) neither positively nor negatively influenced the overall achievement of the experimental group (1969, p. 227).

A need for flexibility of grouping was found. Various measures used to try to predict group membership were shown to be ineffective. Also reported was the fact that during Phase II, mean proportions of basic and intermediate level objectives achieved by students in the basic and intermediate groups increased from .42 to .72 and .52 to .82 respectively. This indicated substantial achievement gains during the remedial activities of Phase II. Mortlock

felt that these findings indicated that "if grouped instruction based on achievement of objectives of the type written is used, then group membership should be flexible with opportunity provided for regrouping for different topics" (1969, p. 228).

Mortlock concluded that the behaviorial objectives met most claims made for them and the students had a very favorable reaction to the two-phase form of classroom management.

Bierden. Bierden (1968) made a companion study to Mortlock on provisions for individual differences in seventh grade mathematics based on grouping and behaviorial objectives. The main procedures in his study were similar to those used by Mortlock. The opportunity for grouping based on the achievement of behaviorial objectives was the main provision for individual differences. Students in neither of the Bierden nor Mortlock studies worked independently on self-study materials.

In the Bierden study, upon completion of the second post-test, at the end of Phase II, the students started Phase III. This involved working on unachieved objectives from this topic or previous topics. The teacher helped individual students in Phase III, and upon completion of all required work, the students did enrichment activities.

Findings from the Bierden study which are relevant

to the present study include:

(a) A need for flexibility of grouping.

Even though it was found that group membership could be determined from scores of I.Q., California Achievement Test pre-test scores (computational), SMSG Mathematics Inventory pre-test scores (concepts) and pre-attitude toward mathematics scores, it was concluded that flexible grouping was still desired as a means of allowing students to change to different groups as their abilities and needs in various topics changed.

(b) The average proportion of objectives achieved by the basic and intermediate groups increased from the pre-test to the post-test, and then decreased on the retention test. However, the proportion on the retention test was higher than on the pre-test. The proportions for the advanced group showed an increase in achievement between the pre-test and the post-test and between the post-test and the retention-test.

The study made no provision for comparisons between the experimental treatment and other grade seven mathematics courses taught at the same time. However, to provide further evaluation, comparisons were made at the end of the year with four groups of grade seven students used in previous studies at the University of Michigan school (1968, p. 96). The results supported the conclusion that the experimental treatment had no adverse effects on the increase

in computational skills measured by the grade placement norms of the California Achievement Test. Also, the experimental treatment had no adverse effects on the increase of knowledge of mathematical concepts measured by the SMSG Mathematics Inventory.

The experimental method also resulted in positive changes in mathematics attitude, and as it also provided a structure for meeting individual differences of students, it was concluded that the experimental method was more effective than other techniques used in the mathematics classroom.

Summary

The studies reviewed indicated some of the various ways of developing programs to meet the needs of individual students. Evaluations of these projects with respect to pupil achievement were reviewed. The results of the studies indicated that the experimental method (providing some form of individualized instruction) either had no adverse effects on pupil achievement, or that students receiving the experimental method scored better on achievement tests than students who did not receive it.

The attitude towards mathematics improved for students receiving the experimental method of instruction. In general, the students liked the type of individualized instruction which they received. Although the pupil

achievement was not always obviously enhanced by the individualized instruction, other advantages such as improved attitude toward mathematics indicate that individualized instruction of some form will continue to receive emphasis from educators.

> While no one line summary can characterize individualized instruction at this time, you may be assured that individualized instruction of some form is a coming certainty (Edling, 1970, p. 16).

IV. SELECTION AND CONSTRUCTION OF TESTS

The written test is not the only form a test may take, however Grobman states that "Perhaps the written test is the most frequently used device in systematic evaluation" (1968, p. 63). One major limitation of tests is that they measure only what the student can do in the testing situation. This is not necessarily what the student can do in an out-of-test situation. Subject matter tests are often limited to measuring knowledge and applying knowledge, but problem-solving abilities and creative abilities can also be measured.

It is not a simple matter to select suitable achievement tests for an assessment of pupil achievement in a particular project. Sometimes it is necessary to construct a test which will be relevant to a given curriculum. Grobman (1968) listed several criteria for the selection and construction of tests. Some of these are discussed below.

Validity and Reliability

Validity refers to a test measuring what it was supposed to measure. A test developed for one curriculum might not be suitable for use in a different curriculum. The test should reflect the aims of the new program.

Reliability refers to consistency. Students who wrote a test would obtain the same score if they wrote the test a second time if the test was perfectly reliable.

Fremont (1969) indicates that one of the main differences between teacher-made and standardized tests is that the standardized test has undergone procedures designed to determine its reliability and validity.

Validity is measured against some other criterion, a criterion always open to question. The criterion could be a grade score, aptitude or ability scores. The correlation between the test scores and the criterion scores is the coefficient of validity. Fremont states "For a rule of thumb, validity coefficients that are considered satisfactory will run from about 0.40 to 0.65" (1969, p. 553). Content or face validity is determined by referring the test questions to a panel of judges or experts who will render an opinion as to whether or not a test item appears to test for that which it was intended.

Reliability is concerned with the consistency of the test. Standardized tests usually require a reliability coefficient of not less than 0.90, while teacher-made tests

may be expected to demonstrate reliability coefficients between 0.60 and 0.80 (Fremont, 1969, p. 554).

Both the validity and reliability of the testing instrument must be determined before deciding to use it, and both are very important in selecting or constructing a test. But, as Cronbach pointed out, "An accurate test is not of much value if it gives precise information about an irrelevant quantity" (1963, p. 212).

Test Grids

Grobman (1968) suggested the use of test grids when selecting or constructing a test. It may be difficult to decide how appropriate or how valid a test actually is with respect to a given curriculum. Two problems encountered when scrutinizing a test or when constructing a new one are balance among subject areas and balance among cognitive skills. A test plan or test grid is useful when analyzing the test. The test grid could have two axes, one for content coverage and one for cognitive skills tested. Test questions could then be assigned to the various categories.

Reading Level

Grobman (1968) also discussed the reading level of tests. If the reading level of the test is above that of the student, the test scores of the student may merely reflect their ability or inability to work through intricate sentences to locate the question being asked. Unless a test

is intended to measure reading skills, it has been suggested that the reading level of the test should be at least one and possibly two years below the average reading level of the grade for which it is being used.

Difficulty Level

The difficulty level of the items on a test depends on the purpose of the test. These purposes are discussed by Grobman (1968). If a test is to serve as a basis for grades, some specialists in psychological measurement suggest that items should have a difficulty level of fifty percent. However, if the test is designed to measure mastery of skills required for future learning, there must be some items which are achieved by all.

Reasons for Constructing Tests

Grobman (1968) states that although it is very difficult to construct a test, most projects have to do it sooner or later. Romberg and Wilson give six reasons why commercially prepared tests may not be appropriate to use in a project and tests would have to be constructed.

> (1) The standardized tests do not cover some important components of mathematical ability.
> (2) The standardized tests tend to 'over-test' some components.
> (3) The standardized tests tend to have been developed with the view of testing a unitary trait.
> (4) The standardized tests often include items for testing components of little importance.
> (5) The standardized tests are too long, from one to three hours.
> (6) The standardized tests are primarily for the

assessment of individuals rather than groups.

Summary

The use of the testing instrument is one of the most important ways to determine the effectiveness of a program with respect to pupil achievement. Attention must be given to the selection or construction and the administration of the testing instrument. Grobman indicates the difficulty of obtaining appropriate tests in the following statement.

> In most cases, however, projects will have difficulty finding suitable achievement and attitude tests to measure some facets of their concern. Developing a new test is still more difficult, but may be necessary. In the choice of an existing test or in the development of a new one, suitability to the curriculum and the test population are of paramount importance (1968, p. 65).

CHAPTER III

THE EXPERIMENTAL DESIGN AND TEST CONSTRUCTION

The present study was a part of the Hardisty Project which was concerned with the development and evaluation of a mode of individualized instruction for grade seven mathematics. The purpose of the study was twofold. First was to determine the effectiveness of the individualized mode of instruction with respect to pupil achievement within the cognitive domain. The second purpose was to assess the grouping procedures implemented in the individualized mode of instruction. The group membership of each student was determined by their achievement of behaviorially stated objectives. Group membership was flexible in that it could be changed after each topic was covered.

I. NATURE OF THE SAMPLE

The grade seven mathematics teachers at the Hardisty Junior High School, Edmonton, Alberta, initiated the Hardisty Project by recognizing that some form of individualized instruction should be offered to their students. Thus when the development of the Hardisty Project by the three graduate students and two professors from the University of Alberta began, the nine grade seven classes at the Hardisty Junior High School became the experimental group.

All five grade seven mathematics teachers at Hardisty were involved with using the individualized mode of instruction which was developed.

Investigating the effect of the individualized mode of instruction on pupil achievement required the organization of a comparison study. A control group was selected for the comparison study. All seven grade seven classes at another junior high school in Edmonton formed the control group. The three grade seven mathematics teachers at the school were involved. Regular classroom instruction as had been used in previous years was used with the control group. The experimental and control groups were similar in that they came from adjacent junior high schools in the same part of the city. Both schools were large junior high schools with 272 and 217 grade seven students in the experimental and control schools respectively, at the beginning of the experiment.

II. DESCRIPTION OF THE INSTRUCTIONAL PLAN

A mode of individualized instruction was developed for implementation in the Hardisty Project. The instructional plan is discussed below. A detailed description and rationale of the instructional plan is included in the companion thesis written by M. Westrom. A sample of the self-study materials written for the Project is in Appendix C of this thesis.

All five grade seven mathematics teachers at [illegible] were involved with using the individualized [illegible]

[illegible]

organization of a comparison group. A control group was selected from the common [illegible]. All seven grade seven classes at another junior high school [illegible]

[illegible]

selection [illegible] used with the control group. [illegible] and control groups [illegible] schools in the same [illegible]

findings in the experimental and control schools [illegible]

III. DESCRIPTION OF THE INSTRUCTIONAL [illegible]

A mode of individualized instruction was deve[illegible]

[illegible]

Behaviorial objectives were written for each topic to be covered. These objectives were classified as Basic, Intermediate, and Advanced. Basic objectives were considered necessary to be achieved for further advancement in a particular topic. Intermediate objectives were generally more difficult, included more ideas, and were expected to be achieved by the average student. Some of the Intermediate objectives corresponded to Basic level objectives and simply required more complex performance of the same algorithm discussed in the basic objective. Other Intermediate objectives specified performance with content not considered essential for further advancement in the course, but which was appropriate for average students. Advanced objectives were still more difficult, included advanced ideas, and were expected to be achieved by the superior student.

Each topic was divided into two phases. During Phase I, the students all worked on common material based on Intermediate objectives, and Basic objectives for which there were no corresponding intermediate objectives. The students read a statement of the objective to be achieved and then worked through the development of the ideas involved. The students then attempted a Check Exercise on the objective. This Check Exercise was parallel to the example of the test item given in the statement of the behaviorial objective which illustrated the behavior required

to achieve the objective. If the student met the criterion of achievement given for the Check Exercise he proceeded on to the next objective written for the next concept to be learned. Thus whenever a student achieved an objective, further instruction on that objective was not given. If he did not meet the criterion of achievement stated, the student worked on activities and exercises provided to give further instruction on that objective. When these activities and exercises were completed, he did a second check exercise. If he achieved this second Check Exercise on an objective, the student proceeded to the next objective. If he was not successful, he could consult a reference book, a student-helper or his teacher for further guidance. An indication of the pages in various reference books corresponding to each objective was given at the end of the Check Exercise. The student-helpers were mainly students who were high achievers and were willing to help other students in their room. They were chosen by the teachers, but their participation as a student-helper was voluntary. After obtaining this further guidance, the student continued with the next objective.

At the completion of Phase I, the student wrote Post Test I. This test consisted of an item for each objective studied. The test items were parallel to the Check Exercises and the sample test item given in the objective. The achievement of the behaviorial objectives

to achieve the [illegible]. If the student [illegible] the criterion [illegible]

[illegible]

learned. But whenever a student did not [illegible] a objective,

[illegible]

did not meet the criterion of achievement stated, the student worked on activities and exercises intended to give further instruction on that objective. When these [illegible] and exercises were completed, he [illegible] check test [illegible].

[illegible]

[illegible], the student [illegible] student helper or the teacher for further [illegible]. An indication of [illegible] corresponding to each [illegible]

[illegible]

who were high [illegible] students in the [illegible] room. They were [illegible] the teacher, but their participation as a student helper was voluntary. After obtaining this further guidance, the student [illegible]

on Post Test I determined which group the student would be in during Phase II.

If the student achieved less than fifty percent of the objectives measured on Post Test I, he entered the Basic group. He then received new self-study materials which were concerned only with Basic objectives. He did just what was considered necessary to progress through the course.

If the student achieved between fifty and ninety percent of the objectives on Post Test I, he entered the intermediate group. Here he reviewed the materials used in Phase I which were based on Intermediate objectives and Basic objectives for which there were no Intermediate objectives. New exercises were provided for further practice.

Students who achieved over ninety percent of the objectives on Post Test I entered the Advanced group. They were presented with Advanced objectives and activities which allowed them to delve more deeply into the subject. They also reviewed any Basic or Intermediate level objectives they may not have achieved.

When the materials in Phase II were completed, Post Test II was written, but only on objectives not achieved on Post Test I.

At the completion of Phase II, problem-solving activities were provided. These were referred to as

'Challengers'. They were non-routine problems requiring the students to use the higher levels of the thinking process. The problems were arranged in sections of increasing difficulty. Only the more able students were expected to try all the problems given.

As this type of instruction allowed the students to progress at different rates of speed, enrichment activities were provided for those who progressed more rapidly. These activities were in the form of laboratory exercises, hard problems, reading selections and mathematical games. The enrichment activities were generally tried only by the Advanced group.

After attempting the Challengers and enrichment activities (for those involved), all students started the next topic at the Intermediate level. Students were grouped for Phase II of each topic independently of their achievement on previous topics. Thus flexibility of grouping was provided.

To enable the student to keep track of his progress through a topic, a flow chart was provided for each student. The flow chart outlined the various paths a student could take during Phase I. A similar flow chart was given to the students in the Basic group during Phase II.

A record page was started for each student at the completion of Post Test I. This enabled both the student and the teacher to know which objectives were yet to be

[illegible] the students to use the higher levels of the thinking process. The problems were arranged in sections of increasing difficulty. [illegible]

[illegible]

[illegible] activities were provided for those who progressed more rapidly. These activities were in the form of [illegible] exercises, hand problems, reading, selection, and [illegible] games. The [illegible] generally [illegible] by the [illegible] group.

[illegible]

[illegible] achievement on previous topic. [illegible] flexibility of grouping was provided.

To enable the student to keep track of his progress through a topic, a flow chart [illegible] prepared for each student. [illegible]

[illegible]

achieved during Phase II. The objectives achieved on Post Test II and the Challenger attempts and successes were also recorded.

During Phase I, 'mini-lectures' were held. The mini-lectures were large group lectures presented to those students who chose to attend them. The content to be covered each period was made known to the students. The content covered during the mini-lectures progressed at the same rate as the maximum time allowed for students to complete Phase I. Some students attended the mini-lectures regularly, while most attended when they needed further instruction on a particular objective.

The individualized mode of instruction developed allowed the students to progress through the materials at varying rates of speed. Due to practical reasons, a maximum time limit was set for the completion of a topic. The time limit gave even the slower students sufficient time to complete the topic. Through the grouping procedures, the mode of instruction allowed the students to work at difficulty levels appropriate for their abilities. The flexibility of grouping provided for by the mode of instruction prevented the placement of students in a particular group for an entire year. Student placement in a group in Phase II depended upon their achievement on that topic in Phase I, irrespective of previous grouping or achievement. A diagram of the instructional plan follows.

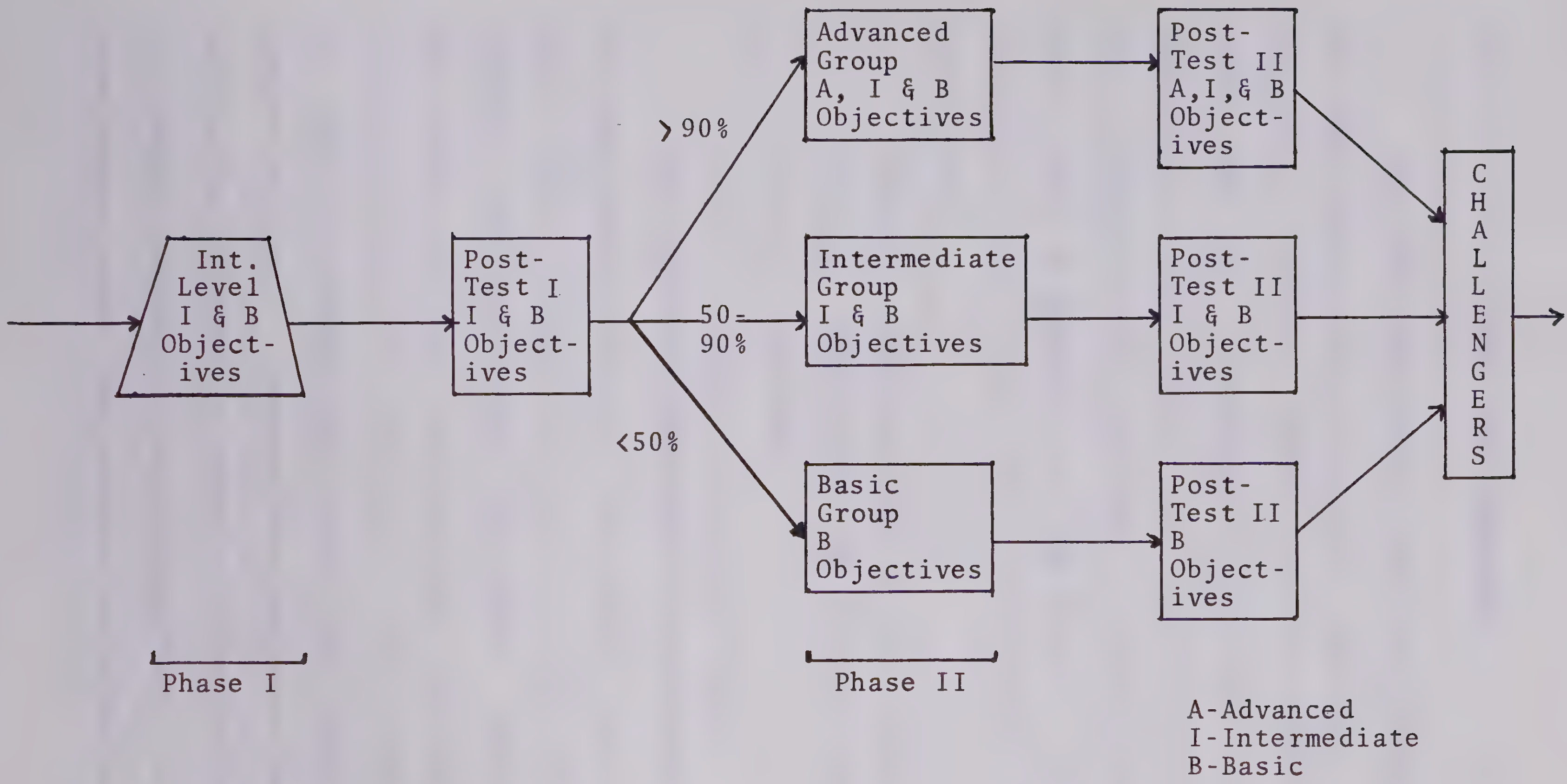

Figure 1

INSTRUCTIONAL PLAN

III. THE TREATMENT

Both the experimental and control groups covered the same subject matter during the experiment. This consisted of rational numbers, rates and percent at the grade seven level. The general objectives for grade seven mathematics in this content area from the Alberta Department of Education were followed. A list of these objectives may be found in Appendix A of the thesis. The content was divided into four topics for the experimental group. These topics were:

Topic 1: Rational Numbers and Fractions;

Topic 2: Operations with Rational Numbers;

Topic 3: Decimals;

Topic 4: Rates, Ratios and Percent.

The individualized mode of instruction was used by the experimental group for five months, from February to the end of June. During this time the content area of rational numbers, rates and percent was studied. However, due to a time factor, the self-study materials developed for the Project were not used for Topic 4. The teachers resorted to a type of extended mini-lecture when presenting the content of Topic 4. Because the individualized mode of instruction attempted to cater to differing learning styles by offering the mini-lecture during Phase I of the first three topics rather than enforce use of the self-study

the same subject matter [illegible]

[illegible] rational [illegible]

[illegible]

may be found in Appendix A of the thesis. The content was

divided into [illegible] topics for [illegible]

[illegible]

The [illegible] individualized mode of instruction [illegible]

[illegible]

materials, it was not considered completely detrimental to the experiment that the self-study materials were not used for Topic 4. It was indicated by the teachers that most students appreciated the change in instructional procedures.

The time taken by the control group to cover the same content area as that covered by the experimental group was approximately one month less than the time required by the experimental group. The decision was made to give the control group the final achievement post-tests as soon as they finished the required content area, because if they had waited until the experimental group was finished, the control group post-tests could have been regarded more as retention tests.

IV. SELECTION AND DESCRIPTION OF TESTS

One of the purposes of this study was to evaluate the pupil achievement of students receiving the mode of individualized instruction developed. A comparison of pupil achievement between the experimental and control group was thus performed. In order to obtain data to perform this comparison, tests were administered to the students.

A battery of pre-tests was administered to both groups at the beginning of the experiment. The battery consisted of a standardized mathematics test, a mathematics achievement test constructed by the writer, and an attitude

materials, it was not considered [illegible]

to the consultant that the self-study materials were not

[illegible]

most students appreciated the [illegible]

[illegible]

[illegible]

the control class [illegible]

[illegible]

tal [illegible]

retention [illegible].

[illegible]

Due to the purpose of this [illegible]

[illegible]

individualized instruction [illegible]

pupil achievement [illegible] group

[illegible]

this conclusion, [illegible]

[illegible]

toward mathematics test. Parallel forms of the achievement tests were administered as post-tests at the completion of the experiment.

The standardized mathematics test selected was the Cooperative Mathematics Test - Arithmetic Form A for the pre-test; and the parallel test, the Cooperative Mathematics Test - Arithmetic Form B was selected for the post-test. They measured achievement in terms of the students comprehension of the basic concepts, techniques and unifying principles in each content area. A major factor considered when selecting these tests was that they included several items on rational numbers, rates and percent. This was the content area covered during the experiment. An overall assessment of the general mathematics achievement of the students was made from the results of these tests.

Parallel mathematics achievement tests referred to as Mathematics Achievement Test Form I and II were constructed by the writer to be used as pre- and post-tests respectively. They were designed to test the achievement of the students on the specific content area covered during the experiment. A description of the construction of these tests is given at the end of this chapter.

The attitude test chosen was the Mathematical Attitude Scale, the final version of the attitude scale designed by Remai (1965) which he used in his study on student attitudes toward mathematics. The attitude test

provided an attitude score to be used to predict group membership in connection with Question 4 which is discussed at the end of Chapter III.

Samples of the mathematics achievement tests constructed by the writer and the attitude scale are in Appendix D of the thesis.

V. DESCRIPTION OF TEST ADMINISTRATION

All pre-tests were administered to both groups during the first week of the experiment. This included the Mathematical Attitude Scale, the Cooperative Mathematics Test - Arithmetic Form A, and the Mathematics Achievement Test Form I. They were administered by the teachers involved with both groups. There was no time limit on the Mathematical Attitude Scale, as it was easily completed in one class period. A time limit of forty minutes was set for each of the two achievement tests. Students responded to all test items on machine-scored answer sheets which were marked by an optical scorer.

The same regulations were placed on the post-test administrations as for the pre-tests. The control group wrote their post-tests about a month earlier than the experimental group as they had completed all subject areas included in the experiment. The achievement post-tests were written by the experimental group as soon as they had finished all four required topics.

VI. SOURCE OF DATA

Each student in each group was given an identification number. Lorge-Thorndike verbal and non-verbal I.Q. scores and Gates Reading Survey - Form I reading scores were obtained for each student from their cumulative record. Past achievement scores, indicated by mid-term examination marks which were determined by the teachers before the beginning of the experiment and used for report card purposes, were obtained for the students in the experimental group. These scores were key-punched on individual IBM data cards along with pre- and post-test achievement scores for each student.

Further information was gathered on the students in the experimental group. As the individualized mode of instruction allowed the students to enter one of three groups during Phase II of each topic, and group membership could be changed for each topic, it was decided to keep track of which groups each student belonged to for the first three topics. A group membership score was determined for each student receiving the individualized mode of instruction. A group stability score was also formed for each student receiving the mode of instruction to indicate how often his group membership changed. An explanation of how these two scores were determined is given in the thesis in the discussion of statistical

[illegible]

Each student [illegible] given a[illegible]

[illegible] Lorge-Thorndike verbal [illegible]

verbal I.Q. scores and [illegible]

[illegible]

Cumulative record [illegible] elementary school, [illegible]

[illegible]

[illegible]

achievement [illegible]

[illegible]

groups during Phase [illegible] topic, and group means [illegible]

[illegible]

first three topics. [illegible] score was deter-

mined for each student [illegible]

[illegible] instruction. [illegible]

[illegible]

analyses used.

VII. NULL HYPOTHESES AND STATEMENT OF ANALYSES USED

Four main questions were of concern in the study. They are listed with the corresponding hypotheses which were tested to indicate how the questions should be answered. The statistical analyses used to test the hypotheses are also given.

Question 1

Does pupil achievement differ between the pre-test and the post-test under individualized (experimental) instruction; and under regular classroom (control) instruction?

Hypothesis 1(a):

There is no significant difference between mathematics achievement pre-test and post-test scores for the experimental group.

Hypothesis 1(b):

There is no significant difference between mathematics achievement pre-test and post-test scores for the control group.

Hypothesis 1(c):

There is no significant difference between the standardized mathematics pre-test and post-test scores for the experimental group.

Hypothesis 1(d):

There is no significant difference between the standardized mathematics pre-test and post-test scores for the control group.

The above hypotheses were tested by using a correlated t-test. They were tested to determine if both the experimental and control treatment produced significant results with respect to achievement. It was expected that significant results would appear, and the following hypotheses were based on this expectation, as further investigation would be pointless if neither treatment produced significant results.

Question 2

Does pupil achievement differ between the individualized (experimental) and the regular classroom (control) instruction, considering the effects of reading ability and I.Q. level of the students?

Hypothesis 2(a):

> There is no significant difference between group means as measured by the mathematics achievement post-test, using I.Q., reading and mathematics achievement pre-test scores as covariates.

Hypothesis 2(b):

> There is no significant difference between group means as measured by the standardized mathematics post-test, using I.Q., reading and standardized mathematics pre-test scores as covariates.

The above two hypotheses were tested using a one-way analysis of covariance with the indicated variables as covariates. Correlations were found among I.Q., reading and achievement pre- and post-test scores to justify the

correlated t-test. They were tested to determine if both the experimental and control treatment produced a gr[illegible]

[illegible]

hypotheses were based on this expectation [illegible] investigation would be pointless if neither treatment [illegible]

[illegible]

Does pupil achievement differ between the individualized experimental and the regular classroom [illegible] ability and I.Q. level of the students?

Hypothesis 1(a)

There is no significant difference between [illegible] as measured by the [illegible] achievement [illegible] and [illegible] to [illegible] [illegible]

[illegible]

use of the above variables as covariates.

To further investigate the effects of reading levels and I.Q. scores on the treatment, the following hypotheses were tested.

<u>Hypothesis 2(c):</u>

At three reading levels, there is no significant difference between group means as measured by the mathematics achievement post-test, using the mathematics achievement pre-test score as a covariate.

<u>Hypothesis 2(d):</u>

There is no significant interaction between reading levels and modes of instruction as measured by the mathematics achievement post-test.

<u>Hypothesis 2(e):</u>

At three I.Q. levels there is no significant difference between group means as measured by the mathematics achievement post-test, using the mathematics achievement pre-test score as a covariate.

<u>Hypothesis 2(f):</u>

There is no significant interaction between I.Q. levels and modes of instruction as measured by the mathematics achievement post-test.

<u>Hypothesis 2(g):</u>

At three reading levels there is no significant difference between group means as measured by the standardized mathematics post-test, using the standardized mathematics pre-test score as a covariate.

<u>Hypothesis 2(h):</u>

There is no significant interaction between reading levels and modes of instruction as measured by the standardized mathematics post-test.

Hypothesis 2(i):

At three I.Q. levels there is no significant difference between group means as measured by the standardized mathematics post-test, using the standardized mathematics pre-test score as a covariate.

Hypothesis 2(j):

There is no significant interaction between I.Q. levels and modes of instruction as measured by the standardized mathematics post-test.

A study by Bloom (1968) indicated that there is a high correlation of 0.60 between grade point average and reading comprehension. Reading was anticipated to be an especially important learning factor for students in the experimental group, as the self-study materials required more reading from the student than is usual in regular classroom instruction in mathematics. The three reading and the three I.Q. levels were high, medium and low.

The high, medium and low reading levels and I.Q. levels were determined by ranking the students from both groups with respect to reading scores and then I.Q. scores. They were then divided into three equal sections for both scores. The high reading level consisted of students with reading scores over 100, the medium reading level students had reading scores between 87 and 100, and the low reading level had reading scores under 87. The high I.Q. level consisted of students with I.Q. scores over 114, the medium I.Q. level had I.Q. scores between 104.5 and 114, and the low I.Q. level had I.Q. scores under 104.5. The I.Q.

scores used were the average of the verbal and non-verbal I.Q. scores which had been obtained for each student.

The hypotheses were tested by a two-way analysis of covariance, using the indicated variables as covariates.

Question 3

What percentage of objectives are achieved by pupils receiving individualized instruction in each of the three groups at the end of Phase II?

Basic Group. As the Basic objectives studied by the Basic group in Phase II were different from the Intermediate objectives studied during Phase I, a statistical analysis could not be performed. Descriptive results only were obtained. The average percent of Basic objectives achieved by pupils in the Basic group after the completion of Phase II was calculated for each of the first three topics.

Intermediate Group. The Intermediate group studied the same objectives during Phase II as Phase I. Thus, a statistical analysis was used to test the following hypothesis.

Hypothesis 3:

> For the intermediate group, there is no significant difference between the number of objectives achieved after completion of Phase I and after the completion of Phase II for each of the first three topics.

A correlated t-test was used to test the hypothesis.

<u>Advanced Group</u>. Students in the Advanced group studied Advanced objectives during Phase II of each topic. As these were different from the Intermediate objectives studied in Phase I, a statistical analysis could not be used to compare achievement before and after completion of Phase II. Descriptive results only were given, as the percentage of Advanced objectives achieved by the Advanced group were calculated at the end of each of the first three topics.

Question 3 was considered in order to determine if Phase II of the instructional mode was a beneficial part of the mode of instruction developed.

The individualized mode of instruction allowed students to change their group membership for each topic. An investigation was made to determine how many changes were made and if it was significant. An attempt to predict the average group membership of each student was made. Question 4 was formed with respect to the flexibility of grouping procedures discussed above.

<u>Question 4</u>

Is the flexibility of grouping based on the achievement of behaviorial objectives, as provided in the individualized mode of instruction necessary?

<u>Hypothesis 4(a):</u>

There is no significant difference between zero and the mean group stability score of students receiving the individualized mode of instruction.

Advanced Group. Students in the Advanced group [illegible]

[illegible]

percentage of Advanced objectives achieved by the 4th [illegible]

[illegible]

topics.

Question 1 was considered in order to determine if [illegible]

[illegible]

of the mode of instruction [illegible].

The individualized mode of instruction [illegible]

[illegible]

An investigation was made to determine [illegible] changes were made and if it was significant. An attempt to predict the average group membership of each [illegible].

Question 4 was tested with respect to the [illegible]

[illegible]

To test this hypothesis, a group stability score was determined. A value of zero was given if a student stayed in the same group in Phase II for two consecutive topics. If the student changed from Basic to Intermediate, Intermediate to Basic, Intermediate to Advanced or Advanced to Intermediate groups, a value of one was given. If the student moved from Basic to Advanced or Advanced to Basic, a score of two was given. The sum of these values was the group stability score.

A t-test was applied to determine if the mean group stability score was significantly different from zero. A score of zero indicated no group movement.

Hypothesis 4(b):

> There is no identifiable pattern of characteristics (e.g. verbal I.Q., non-verbal I.Q., reading, attitude, past achievement as measured by the grade seven mid-term mathematics examination, standardized mathematics pre-test, and mathematics achievement pre-test scores) to predict the group to which a student receiving the individualized mode of instruction will usually belong.

To test this hypothesis, an average group membership score was determined for each student. A value of one was given when the student was in the Basic group, a value of two when in the Intermediate group, and a value of three was given when the student was in the Advanced group. The average of these values was the average group membership score for each student.

A step-wise linear regression equation was formed,

using verbal I.Q., non-verbal I.Q., reading, attitudes towards mathematics, past achievement as measured by the grade seven mathematics mid-term examination, standardized mathematics pre-test and mathematics achievement pre-test scores as predictor scores, and the average group membership score as the criterion score. This indicated if the average group membership score could be successfully predicted.

IX. TEST CONSTRUCTION

Two mathematics achievement tests were constructed by the writer to test the specific content area covered during the experiment. One test, referred to as the Mathematics Achievement Test Form I, was used as a pre-test at the beginning of the experiment. The other test, referred to as the Mathematics Achievement Test Form II, was used as the post-test.

The tests were designed to be parallel and were based on the objectives set out by the Alberta Department of Education for grade seven mathematics concerning rational numbers, rates and percent. These objectives are listed in Appendix A of the thesis. Form I was constructed and administered before the construction of Form II began. The construction of Form I is discussed first.

Four to six test items were constructed corresponding to each of the twelve objectives. They were classified

according to Bloom's Taxonomy (1956) and were entered on a test grid. The two axes of the grid were objectives and item classification. There were items for each classification except evaluation. Avital and Shettleworth state that in mathematical performance the evaluation category cannot be distinguished as distinct as it may belong in the category of analysis or synthesis (1969, p. 7). The item classification was performed by the writer, a colleague and a University advisor. Two tests were then made from these items. Each test had one, two or three items for each objective being measured. They were administered to 74 and 64 grade eight students. Grade eight students were chosen because grade seven students had not yet covered these topics. An item analysis was performed on both tests. Selecting the best items for each objective from the two tests as determined by the item analysis, a third test was made.

The third test consisted of one to three items for each objective being measured. It was administered to 145 grade eight students and an item analysis was performed. Considering the results from the item analysis, the questions were revised in an appropriate manner. The revised test was then discussed in a seminar class by graduate students and University professors as to content validity and item structure. The recommended revisions were made and the test was then reviewed by the mathematics teachers

at the control school to ensure that all items on the test were appropriate for the control group. The test was then administered as a pre-test for both experimental and control groups. Special caution was taken to ensure that there was no bias in favor of either group in the construction of the test. The test was based on objectives which were common to both the experimental and control groups.

A copy of the Mathematics Achievement Test Form I, the item analysis results and the test grid formed during the construction of Form I is in Appendix D of the thesis.

Form II, used as a post-test, was constructed to be parallel to Form I which was used as a pre-test. During the construction of the test, pilot studies were made to determine which items were acceptable. Two pilot studies were made on the problem-solving items of the test, as it was felt that it would be more difficult to construct parallel items in this classification. The problem-solving items constructed for Form II and the problem-solving items from Form I were given to thirty grade eight students. The items were analyzed and revisions were made to the Form II items which would make them more parallel to the items from Form I. Parallel items were then constructed for all questions on Form I, and both Form I and Form II were administered to two classes of grade eight students. Each student wrote both tests. Half of each class wrote From I and half wrote Form II during the first administration. A

[illegible]

administered as a pre-test for both experimental and control groups. [illegible] no bias in favor of either group in the construction of the test. The test was based on objectives which were [illegible]

[illegible]

the construction of Form I is [illegible]

Form II, used as a post-test, was constructed to be parallel to Form I which was used as a pre-test. [illegible]

[illegible] items in this classification. The problem-solving items [illegible]

[illegible] which would make them more parallel to the [illegible] from Form I. Parallel items were then constructed [illegible]

week later, the students wrote the test they did not write during the first administration. The writing of the alternative form of the test in different orders was done to eliminate the possibility that students might have found the second test written easier than the first, due to recall of items on the first test.

Item analyses were performend on both tests. The Pearson-Product Correlation of the two tests was found to be .7158. The following table, Table II, gives the mean, variance and Kuder-Richardson 20 reliability score for Form I and Form II as was determined by the pilot study.

TABLE II

MEANS, VARIANCES AND KUDER-RICHARDSON 20 RELIABILITY SCORES FOR FORM I AND FORM II - PILOT STUDY

Test	N	Total Raw Score	Mean	Variance	KR-20 Rel.
Form I	60	24	13.88	15.67	.7113
Form II	57	24	14.05	17.66	.7163

Considering the item analyses, revisions were made on some of the items in Form II. The revised test was then used as the post-test. A copy of Form II and the results of the item analysis performed on it when used as the post-test are found in Appendix D of the thesis.

The means, variances, and Kuder-Richardson 20 reliability scores for Form I and Form II when they were administered as the pre- and post-tests are given in Table III.

TABLE III

MEANS, VARIANCES, AND KUDER-RICHARDSON 20 RELIABILITY SCORES FOR FORM I AND FORM II

Test	Group	N	Total Raw Score	Mean	Var.	KR-20 Rel.
Form I	Exp.	257	24	7.23	11.30	.5794
	Control	203	24	8.48	11.02	.5395
Form II	Exp.	258	24	11.81	19.88	.7284
	Control	216	24	10.50	19.06	.7171

CHAPTER IV

THE FINDINGS OF THE INVESTIGATION

In this chapter the findings of the study are presented. Under the four main questions of the study, the data relating to the corresponding hypotheses are presented and analyzed.

The purpose of the study was twofold. First was to assess the effect of the individualized mode of instruction with respect to pupil achievement within the cognitive domain. The second purpose was to determine the effectiveness of the grouping procedures incorporated in the individualized mode of instruction.

The first two main questions of the study are concerned with the first purpose of the study, that of assessing pupil achievement. The results of investigations made concerning the first two questions are now reported.

I. QUESTION 1

Does pupil achievement differ between the pre-test and the post-test under individualized (experimental) instruction; and under regular classroom (control) instruction?

It was considered necessary to determine if both modes of instruction produced significant results with respect to pupil achievement as non-significant results

would indicate that further investigation was pointless. Pupil achievement was measured by the pre- and post mathematics achievement tests and the pre- and post standardized mathematics tests. Only students who had written all four tests were included in the sample. The hypotheses tested to answer this question are given.

Hypothesis 1(a):

There is no significant difference between mathematics achievement pre-test and post-test scores for the experimental group.

Hypothesis 1(b):

There is no significant difference between mathematics achievement pre-test and post-test scores for the control group.

Hypothesis 1(c):

There is no significant difference between the standardized mathematics pre-test and post-test scores for the experimental group.

Hypothesis 1(d):

There is no significant difference between the standardized mathematics pre-test and post-test scores for the control group.

The above hypotheses were tested by using a correlated t-test. The mean ($\overline{X}$) and standard deviation (S.D.) of each test is indicated in Table IV along with the correlations (r) and t-ratios between pre- and post-tests. The total raw score for the standardized mathematics test was 50, and for the mathematics achievement test it was 24.

would indicate that [illegible] investigation was justified.

Pupil achievement [illegible] by the pre- and post-

[illegible]

written and [illegible] were included in the sample.

[illegible]

There is no significant difference [illegible] achievement [illegible] and the control group.

[illegible]

There [illegible] experimental pre-test and post-test [illegible] for the control group.

[illegible]

correlated t-test. The mean ($\bar{X}$) and standard deviation (S.D.) of each test is presented in Table [illegible] along

[illegible]

TABLE IV

MEANS, STANDARD DEVIATIONS, CORRELATIONS AND t-RATIOS FOR PRE- AND POST-TESTS

Group	Test	N	Pre- ($\overline{X}$)	Post- ($\overline{X}$)	Pre- (S.D.)	Post (S.D.)	r	t-ratio
Exp.	Stand. Math.	220	22.62	28.30	6.53	7.68	.61	13.21*
Control	Stand. Math.	176	24.89	29.60	6.49	7.57	.63	10.13*
Exp.	Math. Ach.	220	7.27	11.86	3.32	4.36	.52	17.53*
Control	Math. Ach.	176	8.62	10.94	3.37	4.42	.64	8.92*

* Correlation significant at the .01 level.

As all t-ratios calculated were significant at the .01 level, all four null hypotheses formed under Question 1 were rejected. Significant differences in pupil achievement as measured by the mathematics achievement pre- and post-tests, and the standardized mathematics pre- and post-tests resulted from both modes of instruction.

To further investigate the effect of the individualized mode of instruction on pupil achievement, it was decided to compare the pupil achievement of the experimental and control groups as measured by the standardized mathematics tests and the mathematics achievement tests. Question 2 was formed with respect to this investigation.

II. QUESTION 2

Does pupil achievement differ between the individualized (experimental) and the regular classroom (control) instruction, considering the effects of reading ability and I.Q. level of the student?

Hypothesis 2(a):

> There is no significant difference between group means as measured by the mathematics achievement post-test, using I.Q., reading and mathematics achievement pre-test scores as covariates.

Hypothesis 2(b):

> There is no significant difference between group means as measured by the standardized mathematics post-test, using I.Q., reading and the standardized mathematics pre-test scores as covariates.

The above two hypotheses were tested by a one-way analysis of covariance using I.Q., reading and the indicated pre-test as covariates. The I.Q. score used was the average of the Lorge-Thorndike verbal and non-verbal I.Q. scores. The reading score was obtained for each student from the Gates Reading Survey - Form I. Only students who had written all four achievement tests were included in the sample.

The correlations among the I.Q., reading, mathematics achievement pre- and post-test and the standardized mathematics pre- and post-test scores are given for the experimental and control group in Table V and Table VI respectively.

TABLE V

CORRELATIONS AMONG VERBAL I.Q., NON-VERBAL I.Q., READING, MATHEMATICS ACHIEVEMENT PRE- AND POST-TEST AND STANDARDIZED MATHEMATICS PRE- AND POST-TEST SCORES FOR THE EXPERIMENTAL GROUP

	1 V.I.Q.	2 N.V.I.Q.	3 Reading	4 Math. Ach. Pre	5 Math. Ach. Post	6 Stand. Pre	7 Stand. Post
1	1.00	.75	.76	.44	.45	.61	.57
2		1.00	.57	.47	.48	.62	.54
3			1.00	.41	.48	.57	.58
4				1.00	.52	.63	.45
5					1.00	.59	.67
6						1.00	.61
7							1.00

TABLE VI

CORRELATIONS AMONG VERBAL I.Q., NON-VERBAL I.Q., READING, MATHEMATICS ACHIEVEMENT PRE- AND POST-TESTS AND STANDARDIZED MATHEMATICS PRE- AND POST-TEST SCORES FOR THE CONTROL GROUP

	1 V.I.Q.	2 N.V.I.Q.	3 Reading	4 Math. Ach. Pre	5 Math. Ach. Post	6 Stand. Pre	7 Stand. Post
1	1.00	.75	.72	.47	.48	.61	.55
2		1.00	.57	.44	.44	.52	.54
3			1.00	.40	.46	.53	.61
4				1.00	.62	.57	.55
5					1.00	.61	.67
6						1.00	.61
7							1.00

Analysis of covariance combines the advantages of regression analysis with analysis of variance (Kirk, 1968, p. 455). Through regression analysis the dependent variate can be adjusted so as to remove the effects of the uncontrolled source of variation (covariate). Analysis of variance is then performed on the adjusted scores. The regression equations formed for each group with respect to each achievement test are stated in Table VII. I.Q. (X_1), reading (X_2), mathematics achievement pre-test (X_3) and the standardized mathematics pre-test (X_4) scores are the predictor variables.

TABLE VII

REGRESSION EQUATIONS FORMED DURING ANALYSIS OF COVARIANCE CALCULATIONS INVOLVING THE MATHEMATICS ACHIEVEMENT AND THE STANDARDIZED MATHEMATICS TESTS

Group	Test	Regression Equation
Exp.	Math. Ach.	$\hat{Y} = -4.88 + .82 + .08(X_1) + .04(X_2) + .53(X_3)$
Control	Math. Ach.	$\hat{Y} = -4.88 + -.82 + .08(X_1) + .04(X_2) + .53(X_3)$
Exp.	Stand. Math.	$\hat{Y} = -5.19 + -.11 + .10(X_1) + .13(X_2) + .42(X_4)$
Control	Stand. Math.	$\hat{Y} = -5.19 + .11 + .10(X_1) + .13(X_2) + .42(X_4)$

The analyses of variance, performed on the adjusted means of the mathematics achievement post-test and the standardized mathematics post-test, results are given in Table VIII and Table IX respectively. I.Q., reading and the corresponding pre-tests scores were used as covariates.

TABLE VIII

SUMMARY TABLE FOR ANALYSIS OF COVARIANCE RESULTS INVOLVING THE MATHEMATICS ACHIEVEMENT TEST

Source	SS	DF	MS	F-Ratio	Prob.
A	249.82	1	249.82	21.75	0.00000*
error	4490.26	391	11.48		

* F-ratio significant at the .01 level

TABLE IX

SUMMARY TABLE FOR ANALYSIS OF COVARIANCE RESULTS INVOLVING THE STANDARDIZED MATHEMATICS TEST

Source	SS	DF	MS	F-Ratio	Prob.
A	4.67	1	4.67	.15	0.69
error	11738.5	391	30.02		

Hypothesis 2(a) was rejected as the F-ratio was significant at the .01 level. The experimental group had a significantly higher adjusted group mean than the control

group with respect to pupil achievement as measured by the mathematics achievement post-test, using I.Q., reading, and mathematics achievement pre-test scores as covariates. The adjusted group mean for the experimental group was 1.644 points higher than the adjusted group mean for the control group.

Hypothesis 2(b) was not rejected as the F-ratio was not significant at the .01 level. There was no significant difference between group means as measured by the standardized mathematics post-test, using I.Q., reading, and standardized mathematics pre-test scores as covariates. The adjusted group mean of the control group was .2246 points higher than the experimental adjusted group mean.

To continue the comparison of pupil achievement of the experimental and control groups, the achievement scores of the two groups at three reading levels and three I.Q. levels were compared. Both reading and I.Q. scores were found to be significant predictors of the criterion score in the regression equation formed in the analysis of covariance reported above. This tended to support the use of reading and I.Q. scores as covariates and also the further investigation done pertaining to comparison of pupil achievement at three reading and three I.Q. levels. Reading was expected to be an important learning factor for the experimental group due to the large amount of reading required of these students using the self-study materials.

The three reading levels and the three I.Q. levels were high, medium and low. Students in the high reading level had reading scores over 100, students in the medium level had scores between 87 and 100, while students in the low reading level had scores below 87. Students in the high I.Q. level had I.Q. scores greater than 114, students in the medium level had scores between 104.5 and 114, while students in the low I.Q. level had I.Q. scores less than 104.5.

The means and standard deviations of the verbal I.Q., non-verbal I.Q., and reading scores for each group are stated in Table X.

TABLE X

MEANS AND STANDARD DEVIATIONS OF VERBAL I.Q., NON-VERBAL I.Q., AND READING SCORES

Group	N	Score	Mean	S.D.
Exp.	191	V.I.Q.	109.02	16.21
		N.V.I.Q.	109.18	15.63
		Reading	92.09	16.33
Control	171	V.I.Q.	109.72	13.14
		N.V.I.Q.	109.59	12.28
		Reading	94.39	16.56

Two-way analyses of covariance were performed for the comparison of pupil achievement at three reading levels and three I.Q. levels. Pupil achievement was measured by the mathematics achievement post-test and the standardized mathematics post-test. The corresponding pre-tests were used as covariates.

Analysis of covariance requires that equal or proportionate numbers of scores are in the various subgroups. Lewis suggests that one way to obtain equal or proportionate cell frequencies is to allow more pupils in each group than is actually needed and then randomly reject scores from the unduly large subgroups. If only a few scores have to be rejected, Lewis feels this might be the most sensible method to adopt (1968, p. 180). This procedure was used to obtain proportional cell frequencies in the two-way analysis of covariance used for comparison of pupil achievement.

The null hypotheses tested in this comparison are stated. Following each hypothesis is a table giving the adjusted cell means from the analysis of covariance calculations and a summary table reporting the analysis of covariance results.

Hypothesis 2(c):

At three reading levels, there is no significant difference between group means as measured by the mathematics achievement post-test, using the mathematics achievement pre-test score as a covariate.

Hypothesis 2(d):

There is no significant interaction between reading levels and modes of instruction as measured by the mathematics achievement post-test.

TABLE XI

ADJUSTED CELL MEANS ($\overline{X}$) OF THREE READING LEVELS FOR EXPERIMENTAL AND CONTROL GROUPS INVOLVING MATHEMATICS ACHIEVEMENT POST-TEST RESULTS

Group	High Reading		Medium Reading		Low Reading	
	N	$\overline{X}$	N	$\overline{X}$	N	$\overline{X}$
Exp.	71	13.54	60	12.28	69	10.87
Control	62	11.99	50	9.92	60	9.28

TABLE XII

SUMMARY TABLE FOR ANALYSIS OF COVARIANCE RESULTS INVOLVING MATHEMATICS ACHIEVEMENT POST-TEST SCORES AT THREE READING LEVELS

Source	SS	DF	MS	F-ratio
I (Reading)	423.03	2	211.51	17.83*
J (Treatment)	293.01	1	293.01	24.71*
I x J (Interaction)	9.13	2	4.56	.38
Error	4330.29	365	11.86	

* F-ratio significant at the .01 level

Hypothesis 2(c) was rejected as the F-ratio for treatment effect was significant at the .01 level. At three reading levels, the experimental group had significantly greater adjusted group means as measured by the mathematics achievement post-test, using the mathematics achievement pre-test score as a covariate. To determine which reading levels had significant differences, a one-way analysis of covariance was made at each of the three reading levels. The pupil achievement was measured by the mathematics achievement post-test and the corresponding pre-test was used as a covariate. The results of these analyses are in Table XIII.

TABLE XIII

SUMMARY TABLE FOR ANALYSIS OF COVARIANCE RESULTS FOR EACH READING LEVEL INVOLVING THE MATHEMATICS ACHIEVEMENT POST-TEST

Source	SS	DF	MS	F-ratio	Prob.
A (High Reading)	70.29	1	70.29	5.89	0.017*
Within	1526.40	128	11.93		
A (Medium Reading)	168.51	1	168.51	12.48	0.001**
Within	1471.61	109	13.50		
A (Low Reading)	67.11	1	67.11	6.46	0.012*
Within	1309.64	126	10.39		

* F-ratio significant at the .02 level
** F-ratio significant at the .01 level

[illegible] was selected as the [illegible]

[illegible] effect was significant at the [illegible] level. At

[illegible]

achievement [illegible] test score as a covariate. [illegible]

which reading [illegible]

way analysis of covariance was [illegible]

mathematics achievement [illegible]

pretest was used as a covariate. The results of these

analyses are in Table [illegible]

[illegible]	[illegible]	[illegible]	[illegible]	12.18

The experimental group had significantly higher adjusted means as measured by the mathematics achievement post-test for each of the three reading levels.

Hypothesis 2(d) was not rejected as the F-ratio for the interaction effect (Table XII) was not significant at the .01 level. There was no significant interaction between reading levels and modes of instruction involving the mathematics achievement post-test.

Hypothesis 2(e):

> At three I.Q. levels, there is no significant difference between group means as measured by the mathematics achievement post-test, using the mathematics achievement pre-test scores as a covariate.

Hypothesis 2(f):

> There is no significant interaction between I.Q. levels and modes of instruction as measured by the mathematics achievement post-test.

TABLE XIV

ADJUSTED CELL MEANS ($\overline{X}$) OF THREE I.Q. LEVELS FOR EXPERIMENTAL AND CONTROL GROUPS INVOLVING MATHEMATICS ACHIEVEMENT POST-TEST RESULTS

Group	High I.Q.		Med. I.Q.		Low I.Q.	
	N	$\overline{X}$	N	$\overline{X}$	N	$\overline{X}$
Exp.	64	13.98	71	11.58	71	10.80
Control	55	12.47	59	10.23	62	9.05

TABLE XV

SUMMARY TABLE FOR ANALYSIS OF COVARIANCE RESULTS INVOLVING MATHEMATICS ACHIEVEMENT POST-TEST SCORES AT THREE I.Q. LEVELS

Source	SS	DF	MS	F-ratio
I (I.Q.)	556.83	2	278.42	23.98*
J (Treatment)	214.25	1	214.25	18.45*
I x J (Interaction)	1.72	2	.86	.07
Error	4354.28	375	11.61	

* F-ratio significant at the .01 level

Hypothesis 2(e) was rejected as the F-ratio for treatment effect was significant at the .01 level. At three I.Q. levels, the experimental group had significantly higher group means as measured by the mathematics achievement post-test, using the mathematics achievement pre-test as a covariate. To determine which I.Q. levels had significant differences a one-way analysis of covariance was performed at each of the three I.Q. levels. The mathematics achievement pre-test score was the covariate. The results of these analyses are reported in Table XVI.

The experimental group had significantly higher adjusted group means as measured by the mathematics achievement post-test for each of the three I.Q. levels.

TABLE XVI

SUMMARY TABLE FOR ANALYSIS OF COVARIANCE RESULTS FOR EACH I.Q. LEVEL INVOLVING THE MATHEMATICS ACHIEVEMENT POST-TEST

Source	SS	DF	MS	F-ratio	Prob.
A (High I.Q.)	58.51	1	58.51	5.78	0.018**
Within	1173.92	116	10.12		
A (Med. I.Q.)	76.41	1	76.41	4.97	0.027*
Within	1950.72	127	15.36		
A (Low I.Q.)	89.19	1	89.19	9.71	0.002***
Within	1194.57	130	9.19		

* F-ratio significant at the .03 level
** F-ratio significant at the .02 level
*** F-ratio significant at the .01 level

Hypothesis 2(f) was not rejected as the F-ratio for the interaction effect (Table XV) was not significant at the .01 level. There was no significant interaction between I.Q. levels and mode of instruction involving the mathematics achievement post-test.

<u>Hypothesis 2(g):</u>

At three reading levels, there is no significant difference between group means as measured by the standardized mathematics post-test, using the standardized mathematics pre-test as a covariate.

<u>Hypothesis 2(h):</u>

There is no significant interaction between reading levels and modes of instruction as measured

Hypothesis [illegible] was not rejected [illegible] F-ratio [illegible]

the [illegible] effect [illegible] was not significant at

the .01 level, [illegible]

between [illegible] of instruction [illegible] the

mathematics achievement posttest.

by the standardized mathematics post-test.

TABLE XVII

ADJUSTED CELL MEANS ($\overline{X}$) OF THREE READING LEVELS FOR EXPERIMENTAL AND CONTROL GROUPS INVOLVING THE STANDARDIZED MATHEMATICS POST-TEST RESULTS

Group	High Read.		Med. Read.		Low Read.	
	N	$\overline{X}$	N	$\overline{X}$	N	$\overline{X}$
Exp.	71	31.15	60	29.26	69	25.98
Control	62	31.61	50	29.53	60	25.44

TABLE XVIII

SUMMARY TABLE FOR ANALYSIS OF COVARIANCE RESULTS INVOLVING THE STANDARDIZED MATHEMATICS POST-TEST SCORES AT THREE READING LEVELS

Source	SS	DF	MS	F-ratio
I (Reading)	1671.04	2	835.52	25.51*
J (Treatment)	.06	1	.06	.001
I x J (Interaction)	18.17	2	9.08	.28
Error	11952.33	365	32.75	

* F-ratio significant at the .01 level

Hypothesis 2(g) was not rejected as the F-ratio for treatment effect was not significant at the .01 level. At three reading levels there was no significant difference among group means as measured by the standardized mathematics post-test using the standardized mathematics

pre-test score as a covariate.

Hypothesis 2(h) was not rejected as the F-ratio for the interaction effect was not significant at the .01 level. There was no significant interaction between reading levels and modes of instruction involving the standardized mathematics post-test.

Hypothesis 2(i):

At three I.Q. levels, there is no significant difference between group means as measured by the standardized mathematics post-test, using the standardized mathematics pre-test score as a covariate.

Hypothesis 2(j):

There is no significant interaction between I.Q. levels and modes of instruction as measured by the standardized mathematics post-test.

TABLE XIX

ADJUSTED CELL MEANS ($\overline{X}$) OF THREE I.Q. LEVELS FOR EXPERIMENTAL AND CONTROL GROUPS INVOLVING THE STANDARDIZED MATHEMATICS POST-TEST RESULTS

Group	High I.Q.		Med. I.Q.		Low I.Q.	
	N	$\overline{X}$	N	$\overline{X}$	N	$\overline{X}$
Exp.	64	32.19	71	27.92	71	26.11
Control	55	31.66	59	29.61	62	25.81

TABLE XX

SUMMARY TABLE FOR ANALYSIS OF COVARIANCE RESULTS INVOLVING THE STANDARDIZED MATHEMATICS POST-TEST SCORES AT THREE I.Q. LEVELS

Source	SS	DF	MS	F-ratio
I (I.Q.)	1512.68	2	756.34	22.93*
J (Treatment)	7.65	1	7.65	.23
I x J (Interaction)	94.97	2	47.48	1.44
Error	12369.67	375	32.99	

* F-ratio significant at the .01 level

Hypothesis 2(i) was not rejected as the F-ratio for the treatment effect was not significant at the .01 level. At three I.Q. levels there was no significant difference among group means as measured by the standardized mathematics post-test, using the standardized mathematics pre-test as a covariate.

Hypothesis 2(j) was not rejected as the F-ratio for the interaction effect was not significant at the .01 level. There was no significant interaction between I.Q. levels and modes of instruction, involving the standardized mathematics post-test.

While there was no significant difference between group means of the experimental and control groups with respect to pupil achievement as measured by the standardized mathematics tests, the experimental group had significantly

better results with respect to pupil achievement as measured by the mathematics achievement tests. Further investigation revealed that the experimental group had significantly higher scores than the control group at all three reading and I.Q. levels with respect to pupil achievement as measured by the mathematics achievement tests.

There was no significant interaction effect between reading or I.Q. levels and modes of instruction.

The second purpose of the study involved an intrinsic evaluation of the individualized mode of instruction with respect to the grouping procedures used. Questions 3 and 4 were concerned with this aspect of the study. The results of investigations concerning these questions are now reported.

According to the instructional plan for the individualized mode of instruction, each topic was divided into two phases. Students worked through Phase I at the Intermediate level, and depending upon their achievement of the behavioral objectives studied in Phase I, they entered either the Basic, Intermediate or Advanced group for Phase II. Phase II allowed the students in the Basic group to concentrate on Basic objectives, ones that were considered necessary to be achieved for further progress through the course. Students in the Intermediate group reviewed their unachieved objectives from Phase I.

Advanced objectives were presented to the students in the Advanced group. Thus as well as reviewing any unachieved objectives from Phase I, they were able to delve more deeply into the subject matter. It was of interest to determine how effective Phase II was. How many Basic objectives were the Basic group able to master? Did the Intermediate group achieve significantly more Intermediate objectives after Phase II than at the beginning of Phase II? How many Advanced objectives did the Advanced group achieve? These questions were combined to form Question 3.

III. QUESTION 3

What percentage of objectives are achieved by pupils receiving individualized instruction in each of the three groups at the end of Phase II?

Basic Group. As the Basic objectives studied during Phase II were different from the Intermediate objectives studied during Phase I, descriptive results only are given. No statistical analyses were performed. The percentage of Basic objectives achieved by the Basic group after the completion of Phase II for each of the first three topics was calculated. These findings are reported in Table XXI.

TABLE XXI

PERCENT OF BASIC OBJECTIVES ACHIEVED BY THE BASIC GROUP AT THE END OF PHASE II FOR EACH TOPIC

Topic	N	Ave. Percent	S.D.
1	104	70.77	17.44
2	75	65.70	20.66
3	111	53.96	22.05

Students who were in the Basic group because they had achieved less than fifty percent of the objectives studied in Phase I, achieved an average of 63.28 percent of the Basic objectives they studied in Phase II.

Intermediate Group. A comparison between the average number of objectives achieved by the Intermediate group after the completion of Phase I and Phase II was made for each of the first three topics. As the same objectives were studied during both Phases by the Intermediate group, a statistical analysis could be performed. The following hypothesis was tested:

Hypothesis 3:

For the Intermediate group, there is no significant difference between the number of objectives achieved after the completion of Phase I and Phase II, for each of the first three topics.

A correlated t-test for significance of difference between means was used to test this hypothesis. Table XXII

shows the average number and average percent of objectives achieved after the completion of Phase I and Phase II, the standard deviations, the correlations and t-ratios for each topic.

TABLE XXII

AVERAGE NUMBER AND PERCENT OF INTERMEDIATE OBJECTIVES ACHIEVED AFTER PHASE I AND PHASE II, STANDARD DEVIATIONS, CORRELATIONS AND t-RATIOS

Topic	N	Phase	Ave. No.	Ave. Percent	S.D.	r	t-ratio
1	98	I	11.58	60.90	1.67	.31	27.48*
		II	16.83	88.56	1.53		
2	145	I	10.28	60.47	1.73	.69	25.59*
		II	13.83	81.40	2.29		
3	125	I	8.67	64.50	1.79	.61	20.59*
		II	11.95	79.70	2.14		

* t-ratio significant at the .01 level

Hypothesis 3 was rejected for all three topics as all three t-ratios were significant at the .01 level. There was a significant difference between the average number of Intermediate objectives achieved after Phase I and after Phase II for each of the first three topics by the Intermediate group. Students in the Intermediate group achieved an average, on all three topics, of 61.96 percent of the Intermediate objectives after Phase I and 83.22 percent of the Intermediate objectives after reviewing

the unachieved ones from Phase I during Phase II.

Advanced Group. Students in the Advanced group during Phase II had achieved over ninety percent of the Intermediate objectives studied in Phase I. As they had only a few unachieved objectives to review, most of their time was spent studying the Advanced objectives. As these Advanced objectives were different from the Intermediate objectives studied in Phase I, a statistical analysis was not done. Descriptive results of the percentage of Advanced objectives achieved by the Advanced group at the completion of Phase II for each of the first three topics are given in Table XXIII.

TABLE XXIII

PERCENT OF ADVANCED OBJECTIVES ACHIEVED BY THE ADVANCED GROUP AT THE END OF PHASE II FOR EACH TOPIC

Topic	N	Ave. Percent	S.D.
1	60	72.72	26.01
2	43	58.49	41.53
3	29	25.37	29.73

The percent of students in each of the three groups for each topic is given in Table XXIV.

TABLE XXIV

PERCENT OF STUDENTS IN EACH OF THE THREE GROUPS FOR EACH TOPIC

Group	Topic 1	Topic 2	Topic 3	Ave.
Basic	40	28	42	37
Intermediate	37	55	48	47
Advanced	23	17	10	16

The findings shown in Table XXII, and Table XXIV indicate that an average of 47 percent of the students (those in the Intermediate group) had achieved an average of 83 percent of the Intermediate objectives after Phase II. An average of 17 percent of the students who were in the Advanced group had already achieved 90 percent of the Intermediate objectives after Phase II for each topic. Thus, on the average, 63 percent of the students had mastered an average of 83 percent or more of all Intermediate objectives after Phase II for all three topics.

Students receiving the individualized mode of instruction were able to change group membership (Basic, Intermediate or Advanced) at the end of each topic. This provided flexibility of grouping. Question 4 was formed concerning this aspect of the instructional model.

IV. QUESTION 4

Is the flexibility of grouping based on achievement of behaviorial objectives provided for in the individualized mode of instruction necessary?

The first investigation performed concerning this question was to determine the amount of movement that existed from students changing from one group to another for two successive topics. A group stability score was thus determined for each student. A value of zero was given if a student stayed in the same group in Phase II for two consecutive topics. If the student changed from the Basic to Intermediate groups; Intermediate to Basic, Intermediate to Advanced, or the Advanced to Intermediate group, a value of one was given. If he moved from the Basic to Advanced, or Advanced to Basic group, a score of two was given. The sum of these values was the group stability score for each student. The maximum group stability score possible was four, and the minimum group stability score was zero. A group stability score of zero would have indicated that the student had not changed from one group to another for any topic.

The number of students who never changed group membership, who changed once and the number of students who changed group membership twice for the three topics covered is given in Table XXV.

TABLE XXV

NUMBER OF STUDENTS CHANGING GROUPS FOR SUCCESSIVE TOPICS

N	Never	Once	Twice
249	106	92	51

These findings indicate that over half of the students changed group membership at least once and that twenty percent of the students changed group membership twice for the three topics covered in this manner. The following hypothesis was tested to determine if the group stability score determined for each student was significantly different from zero.

Hypothesis 4(a):

There is no significant difference between zero and the mean group stability score of students receiving the individualized mode of instruction.

A t-test (McNemar, 1962, p. 101) was applied to test this hypothesis. Table XXVI indicates the mean, standard deviation and t-ratio of the group stability score.

TABLE XXVI

MEAN, STANDARD DEVIATION AND t-RATIO FOR THE GROUP STABILITY SCORE

N	Mean	S.D.	t-ratio
249	.78	.76	16.11*

* t-ratio significant at the .01 level

Hypothesis 4(a) was rejected as the t-ratio was significant at the .01 level. The mean group stability score was significantly different from zero. This indicated there was significant flexibility in group membership among the students.

A further investigation was made with respect to the flexibility of grouping provided for in the individualized mode of instruction. An attempt was made to predict the average group membership of each student. The following hypothesis was tested in this investigation.

Hypothesis 4(b):

> There is no identifiable pattern of characteristics (e.g. verbal I.Q., non-verbal I.Q., reading, attitude toward mathematics, past achievement as measured by the grade seven mid-term examination, standardized mathematics pre-test and mathematics achievement pre-test scores) to predict the group to which a student receiving the individualized mode of instruction will usually belong.

An average group membership score was determined for use as the criterion score in the step-wise linear regression equation formed to test the hypothesis. A value of one was given when the student was in the Basic group, a value of two when in the Intermediate group, and a value of three when the student was in the Advanced group. The average of these values was the average group membership score for each student.

The following scores were used as predictor variables in the regression equation.

(1) V.I.Q.: Lorge-Thorndike Verbal I.Q. score;

(2) N.V.I.Q.: Lorge-Thorndike Non-verbal I.Q. score;

(3) Reading: Gates Reading Survey - Form I reading score;

(4) Past: A past achievement score measured by the grade seven mid-term examination which was determined by the teacher and used for report card purposes;

(5) Att.: attitude to mathematics score as measured by the Mathematics Attitude Scale, where a high score indicates a favorable attitude with a maximum score of 120;

(6) Stand.: raw score on the standardized mathematics pre-test;

(7) Ach.: raw score on the mathematics achievement pre-test.

Table XXVII indicates the means and standard deviations of each of the predictor variables and the criterion variable, and the correlation between each variable and the average group membership score (criterion).

Table XXVII presents the mean [illegible]

each of the [illegible]

and the [illegible]

group [illegible] score ('G' factor).

TABLE XXVII

MEANS, STANDARD DEVIATIONS, AND CORRELATIONS BETWEEN THE VARIABLES AND THE AVERAGE GROUP MEMBERSHIP SCORE

Variable	Mean	S.D.	Correlation (Grp. Mem.)
V.I.Q.	110.15	13.607	.410
N.V.I.Q.	110.63	13.383	.514
Read.	93.53	15.261	.396
Past.	66.81	14.394	.794
Att.	85.13	10.906	.238
Stand.	23.29	6.478	.583
Ach.	7.17	3.352	.467
Grp. Mem.	1.858	5.902	1.000

Table XXVIII gives the statistical results for the prediction of the average group membership score of each student.

TABLE XXVIII

STATISTICAL RESULTS FOR THE PREDICTION OF AVERAGE GROUP MEMBERSHIP SCORES USING SEVEN PREDICTORS

Step	Variable Entering	Prob. Level	Percentage Variance Accounted For	S.E.
1	Past	0.0*	62.97	3.61
2	Stand	0.04	63.80	3.58
3	V.I.Q.	0.05	64.55	3.55
4	N.V.I.Q.	0.04	65.34	3.52
5	Att.	0.09	65.88	3.506
6	Ach.	0.29	66.09	3.505
7	Read.	0.81	66.10	3.51

* Significant at the .01 level

The best fitting equation for predicting average group membership: $Y_{\text{Ave. Grp. Mem.}} = -.3213 + .033\ (\text{Past})$

Past achievement was the only significant predictor of average group membership. The amount of criterion variance accounted for by the inclusion of other predictor variables, over and above that accounted for by past achievement, was not significant at the .01 level. Because of the low standard error of prediction and the significant amount of criterion variance accounted for by the Past Achievement score, Hypothesis 4(b) was rejected. The average group membership of each student could have been predicted by the Past Achievement score.

CHAPTER V

CONCLUSIONS AND IMPLICATIONS

I. THE STUDY

A major emphasis on meeting the needs of individual students is currently being placed on the instructional procedures used in today's schools. Recognizing the need to provide for individual differences, the Hardisty Project developed an individualized mode of instruction for grade seven mathematics, implemented at the Hardisty Junior High School, Edmonton. The mode of instruction that was developed allowed students to progress at varying rates and to study content at varying difficulty levels in accordance with their needs and ability.

Behaviorial objectives were written for all content covered at three instructional levels, Basic, Intermediate and Advanced. Each topic covered was divided into two phases, Phase I and Phase II. During Phase I all students studied Intermediate level objectives. Self-study materials were provided for their use, or if the students so chose, they could attend mini-lectures which included the same content covered in the self-study materials. At the completion of Phase I, the students wrote a diagnostic post-test, Post Test I, on all Intermediate objectives studied. Their achievement on these objectives determined

which group they would enter for Phase II of the topic. If the students achieved less than fifty percent of the Intermediate objectives studied, they entered the Basic group for Phase II. Here they concentrated on the mastery of Basic objectives, ones that were considered essential for further progression through the course. If the students achieved between fifty and ninety percent of the Intermediate objectives studied in Phase I, they reviewed the unachieved Intermediate objectives. These students were in the Intermediate group. Students achieving over ninety percent of the Intermediate objectives entered the Advanced group for Phase II where they reviewed any unachieved objectives from Phase I, and studied Advanced objectives for the topic which allowed them to delve more deeply into the subject.

At the completion of Phase II all students wrote a second post-test, Post Test II, on objectives not achieved on Post Test I. The students then attempted problem-solving exercises referred to as Challengers. The Challengers were of varying difficulty levels so that all students would experience a challenge while solving them. After attempting the Challengers, students who had progressed more rapidly than others did enrichment activities. When Phase I of the next topic was started, all students were again working at the Intermediate level, and their achievement of these objectives would indicate

their group membership for Phase II. Flexibility of grouping was provided for as group membership could be changed for each topic.

Three theses were written concerning the Hardisty Project. One thesis, written by M. Westrom, included the rationale for the many ideas implemented in the mode of instruction and the feasibility of implementing such a program. A thesis written by B. G. te Kampe described the role of the teacher using the individualized mode of instruction and compared it to the role of the teacher using regular classroom instruction. The purpose of the present study was to determine the effectiveness of the individualized mode of instruction with respect to pupil achievement, and to assess the grouping procedures used in the mode of instruction developed.

The investigation into the effects of the mode of instruction on pupil achievement required the forming of a comparison study when the achievement of pupils receiving the individualized (experimental) mode of instruction was compared with achievement of pupils receiving regular classroom (control) instruction. The experimental group consisted of the nine grade seven mathematics classes and the five grade seven mathematics teachers at the Hardisty Junior High School, Edmonton. The seven grade seven mathematics classes and the three grade seven mathematics teachers at another junior high school in the city formed

the control group.

II. DISCUSSIONS AND CONCLUSIONS

Pupil Achievement

It was determined that both the individualized (experimental) mode of instruction and the regular classroom (control) instruction produced significant results with respect to pupil achievement as measured by the mathematics achievement pre- and post-tests and the standardized mathematics pre- and post-tests. To further investigate the effect of the individualized instruction on pupil achievement, the achievement of pupils in the experimental group was compared with the pupil achievement of the control group. Pupil achievement was measured by the mathematics achievement tests and the standardized mathematics tests. A one-way analysis of covariance, using I.Q., reading and the corresponding achievement pre-test scores as covariates, indicated that there was no significant difference between the experimental and control groups with respect to pupil achievement as measured by the standardized mathematics post-test; but that the pupil achievement, as measured by the mathematics achievement post-test, of the experimental group was significantly better than the pupil achievement of the control group.

Reading was expected to be an important learning factor for students in the experimental group due to the

large amount of reading expected of those using the self-study materials. Pupil achievement of the two groups was compared at three reading and three I.Q. levels. A two-way analysis of covariance, with the corresponding achievement pre-test score used as covariate, showed that there was no significant difference between the pupil achievement of the experimental and control groups at three reading levels nor at three I.Q. levels, when pupil achievement was measured by the standardized mathematics post-test. The experimental group had significantly better pupil achievement at all three reading levels and at all three I.Q. levels when pupil achievement was measured by the Mathematics Achievement test. Contrary to expectations, there was no interaction effect between the reading level of the student and the mode of instruction used. The achievement of students from the experimental group in the low reading level, as well as the medium and high reading levels, was as good as or better than the achievement of students receiving regular classroom instruction, even though the self-study materials used by the students in the experimental group required them to read more than is usually done in a regular mathematics classroom. While no data was kept on which students attended the mini-lectures, it could be postulated that many of the poor readers did attend, and thus the reading factor was minimized. There was also no significant interaction between I.Q. levels and

modes of instruction.

Most studies made on individualized instruction have found that there is no difference between achievement of students receiving individualized modes of instruction and of students receiving other modes of instruction, or that the achievement of students receiving individualized instruction is higher than the achievement of students using other modes of instruction. Before conclusions can be made as to the effectiveness of the individualized mode of instruction developed by the Hardisty Project with respect to pupil achievement, several factors must be considered.

The standardized mathematics tests chosen had several items corresponding to objectives studied during the experiment. They also included items on objectives that were studied previous to the experiment, and they were intended to test general mathematical concepts. The mathematics achievement tests were constructed to test the specific content area covered during the experiment and was based on objectives common to both modes of instruction. Thus, while the standardized tests had higher reliability scores, the mathematics achievement test was more relevant to the experiment.

The experimental group took about a month longer to complete the required content area than the control group. This can partially be explained by the fact that

modes of instruction.

[illegible]

have found that there is [illegible] of students receiving [illegible]

[illegible]

modes of instruction [illegible]

[illegible]

The standardized achievement [illegible]

[illegible]

were [illegible] mathematics [illegible] based on objectives [illegible]

[illegible]

the individualized mode of instruction was new to both the students and the teachers in the experimental group. It could be postulated that when both the students and teachers gain experience they will progress more rapidly. The extra time taken by the experimental group to complete the required content could also be explained by the composition of the self-study materials. The objectives were dealt with in greater detail and more ideas were presented than is usual in regular classroom instruction.

The time shortage faced by the experimental group meant that the self-study materials prepared for Topic 4 were not used by the students. The objectives were presented to the students through mini-lectures. Because the individualized mode of instruction attempted to cater to differing learning styles, self-study or mini-lecture, the use of the mini-lecture style for all of Topic 4 was not considered completely detrimental to the results of the project. The teachers involved indicated that most students appreciated the change in instructional procedures.

The conclusion formed is that although the individualized mode of instruction required more time than regular classroom instruction, the individualized mode of instruction had a positive effect on pupil achievement. The achievement of students receiving individualized instruction equalled or excelled that of the students

receiving regular classroom instruction. As the individualized mode of instruction allowed students to progress at varying rates, faster students had the opportunity to achieve enrichment objectives; and as the mode of instruction allowed students to work at varying difficulty levels, superior students had the opportunity to achieve Advanced objectives. The enrichment and Advanced objectives were achieved in addition to those objectives that were common to both modes of instruction. Apart from other factors involved in a mode of instruction such as student and teacher attitudes, the conclusions reached indicate that the individualized mode of instruction is a favorable alternative to regular classroom instruction.

Grouping Procedures

Phase II. According to the instructional plan for the individualized mode of instruction, each topic was divided into two phases, Phase I and Phase II. Students studied Intermediate level objectives during Phase I, and depending on their achievement of these objectives, they entered one of three groups, Basic, Intermediate and Advanced, for further study on objectives relating to their need and ability. Students who achieved less than fifty percent of the Intermediate objectives studied during Phase I entered the Basic group. In the Basic group, students concentrated on the mastery of Basic level

receiving regular classroom instruction. As the ...

alized mode of instruction allowed students to ...

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divided into two phases, Phase I and Phase ... students

studied Intermediate level objectives during Phase I, and

depending on their achievement of these ...

entered one of ...

objectives, ones that were considered prerequisite for further progression throughout the course. Students in the Basic group achieved an average of 63 percent of the Basic objectives studied for all three topics. An average of 37 percent of the students were in the Basic group, and spent approximately one week studying these Basic objectives. They had already been exposed to the corresponding Intermediate objectives, which had a greater difficulty level, during Phase I. Bloom (1968) claims that the top 95 percent of all students can achieve mastery of a subject. But he admits that varying amounts of time are required by different students (a ratio of six to one is given). Although time was shown to be an important factor for the mode of instruction developed, it certainly seems worthwhile to give the students the opportunity to concentrate on and achieve a significant number of Basic objectives. Very often in regular classroom instruction, the Basic student just plugs along trying to keep up with the average student and does not succeed.

Students who achieved between fifty and ninety percent of the Intermediate objectives in Phase I, entered the Intermediate group where they reviewed unachieved Intermediate objectives. A significant difference was found between the number of Intermediate objectives achieved before and after Phase II for the first three topics covered. An average of 62 percent of the Intermediate

objectives for the first three topics were achieved after Phase I, with an average of 83 percent of the objectives being achieved after Phase II. Again the time spent in Phase II reviewing unachieved objectives seems to have been worthwhile as a high degree of mastery resulted for the Intermediate group. The average group membership was 47 percent of the students over all three topics.

Students in the Advanced group of Phase II had already achieved over ninety percent of the Intermediate objectives studied in Phase I. In the Advanced group they reviewed any unachieved objectives they might have, and then studied the Advanced objectives prepared for them. They achieved 73 percent of the Advanced objectives studied during Topic 1, 58 percent of the objectives studied during Topic 2 and 25 percent of the objectives studied during Topic 3. While an average of 16 percent of the students were in the Advanced group for all three topics, only 11 percent were in the Advanced group for Topic 3. The Advanced students achieved only 25 percent of the Advanced objectives for Topic 3. The low membership number as well as the low percentage of objectives achieved can be explained by the fact that students were rushed for time and the particular content covered was fairly difficult. Not all students had sufficient time to attempt all of the Advanced objectives, but as the record pages did not make provision for indicating which

Phase II reviewing unachieved objectives, moved to the

17 percent of the students over all three topics.

Students in the Advanced group of Phase II had

objectives studied in Phase ... the Advanced group they

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of the students were in the Advanced group for all three

Topic 5. The Advanced students achieved only ... percent

of the Advanced objectives for Topic 5. The low ...

... number as well as the low ...

objectives were not attempted, it was assumed that all Advanced students had attempted all Advanced objectives. Phase II was considered beneficial for the Advanced students as they were given the opportunity to study higher level objectives which would not normally be presented in a regular mathematics classroom, and in general the students achieved a significant number of these objectives.

The conclusion drawn from the above comments is that Phase II of the instructional plan, where pupil needs and abilities determine the level of instruction given, contributed significantly to the achievement of the pupils involved. A high degree of mastery was displayed as nearly two-thirds, 63 percent, of the students, Intermediate and Advanced, achieved an average of 83 percent or more of the Intermediate objectives after Phase II. The Basic students were given the opportunity to concentrate on the mastery of Basic objectives, while the Advanced students had the chance to study more advanced ideas than they normally would in a regular mathematics classroom.

Flexibility of Grouping. The individualized mode of instruction developed allowed students to change their group membership for each topic (flexibility of grouping). An investigation was made to determine the amount of changing of group membership that occurred. A group

objectives were not attempted, it was assumed that all [illegible]

[illegible]

Phase II was conducted [illegible] for the Advanced [illegible]

[illegible]

a regular mathematics classroom, and in general the [illegible]

[illegible]

The conclusion drawn from the above comments is that Phase II of the instructional plan, where group learning [illegible] contributed significantly to the achievement of the pupils [illegible] [illegible] and Advanced, achieved an average of [illegible] [illegible]

The Basic students were given the opportunity to [illegible]

Advanced students had the chance to study more advanced ideas than they normally would in a regular mathematics classroom.

[illegible]

stability score, indicating how often group membership was changed, was determined for each student receiving the individualized mode of instruction. The maximum score possible was four, while the minimum score of zero indicated that the individual student made no change in group membership. The average group stability score, found to be significantly different from zero, indicated that a significant amount of change in group membership was displayed by the students. Over half of the students changed group membership at least once for the three topics. Twenty percent of the students changed group membership twice for the three successive topics covered. It would appear that the students made significant use of the opportunity given to change group membership.

To extend the investigation on flexibility of grouping, an attempt was made to predict the average group membership of each student from several predictor variables. The predictor variables were verbal and non-verbal I.Q., reading, attitude towards mathematics, past achievement as measured by the grade seven mid-term examination, mathematics achievement pre-test, and standardized mathematics achievement pre-test scores. It was found that the past achievement score was the only significant predictor of average group membership. The percent of criterion variance accounted for, by the inclusion of further variables, over and above that accounted for by the past

[illegible]

the individualized [illegible] lesson [illegible]. The maximum score possible was four, while the minimum score of zero indicated that the [illegible] no change in group member[illegible]

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significant amount of change in group membership [illegible]

[illegible]

group membership at least once for the [illegible]

[illegible]

grouping, an attempt was made to predict the average group [illegible] for each student from several predictor [illegible]. The predictor variables were verbal and non-verbal I.Q., reading, attitude towards mathematics, past achievement [illegible]

[illegible]

achievement score was not significant.

Although the average group membership could be predicted from past achievement scores, it is not being suggested that students should be pre-grouped at the beginning of the term without a chance to change their group membership. Indications have already been observed that the students did make considerable use of the opportunities provided to change their group membership for each topic, depending on their needs and abilities for that topic. However, while pre-grouping from the prediction of average group membership instead of using the flexible grouping procedure is not recommended, teachers could benefit from the information obtained as a result of average group prediction. Teachers would know, with a limited degree of accuracy, if particular students should, on an average, be in the Basic, Intermediate or Advanced group. This may indicate if certain students are not working to their full capability.

The recommendation of allowing for flexibility of grouping is also upheld by the following observation. The students stated that class members did not care what group other students were in, but they were concerned about their own group membership. The grouping procedures incorporated in the individualized mode of instruction gave the students a chance to change group membership for each topic, and provided a motivation for the students to

Although the average group membership could be predicted from past achievement records, it is not being [illegible] beginning of the term with a chance to change their group membership. Indications have already been given that the students will make considerable use of the opportunities provided to change their group membership for each topic, depending on their needs and abilities for that topic. However, while [illegible] from the prediction of average group membership [illegible]. Using the flexible grouping procedure is not recommended, teachers could benefit from the information obtained as a result of [illegible] limited groups of students. If particular students would on an average, be in the lower, intermediate or advanced group. This may indicate that certain students are not working to their full capacity.

do as well or better on each successive topic.

Considering the factors discussed above, the obvious conclusion is that the flexible grouping procedures incorporated in the individualized mode of instruction were used to advantage by the students. The conclusion reached concurs with the conclusions made by Bierden (1968) and Mortlock (1969) concerning the grouping procedures implemented in their studies. Further reference to grouping procedures is made in the companion thesis written by M. Westrom.

Summary of the Conclusions

(1) Both modes of instruction, the individualized mode of instruction and the regular classroom instruction, produced significant results with respect to pupil achievement.

(2) The achievement of the students receiving the individualized mode of instruction equalled or excelled that of students receiving regular classroom instruction.

(3) The student's reading ability did not significantly affect achievement of students receiving the individualized mode of instruction, as the achievement of students at all reading levels, receiving the individualized mode of instruction equalled or excelled the achievement of students, at comparable reading levels, receiving regular classroom instruction. Similarly the student's

I.Q. level did not significantly affect achievement of students receiving the individualized mode of instruction as the achievement of these students, at all three I.Q. levels, equalled or excelled the achievement of students receiving regular classroom instruction.

(4) The individualized mode of instruction required more time to complete the required content area than the regular classroom instruction.

(5) The high level of mastery of objectives and the opportunity for superior students to study higher level ideas supported the use of grouping procedures, based on achievement of behaviorial objectives, which allowed the needs and abilities of the students to determine the level of instruction received during Phase II of each topic.

(6) The significant number of changes made in the group membership of the students indicated that the flexibility of grouping provided for by the individualized mode of instruction was a worthwhile component of the instructional mode.

III. IMPLICATIONS

Implications for the Classroom

The individualized mode of instruction which was developed for the Hardisty Project could be implemented as it is in any school with two or more classes of the same subject being taught at one time, and it could be used with

subjects other than grade seven mathematics. However, before deciding to use this mode of instruction, the teachers must realize that there are both advantages and disadvantages to the mode. The individualized mode of instruction took longer to cover particular content than the time taken by regular classroom instruction. Behaviorial objectives for all content must be written, and self-study materials are required. The main advantage, as indicated in the study, is the provision for a high level of mastery of objectives at a level corresponding to the needs and abilities of the student for a particular topic. The students can progress at their own rate, within limitations, and learn by a style, self-study or mini-lecture, most appropriate for them for a particular topic.

Modifications could be made to the instructional plan which would provide the student with a chance to choose from even more learning styles. The use of manipulative materials and audiovisual materials could be incorporated in the instructional plan in addition to the self-study materials and mini-lectures already used. This should serve to further improve the effectiveness of the instructional mode with respect to pupil achievement. If it was not possible to cater to various learning styles, the grouping procedures, which were very effective, could still be used in an instructional mode. The use of behaviorial objectives, a lecture-discussion presentation

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instruction took longer to cover particular content than the time taken in regular classroom instruction. [illegible] all content must be [illegible]

[illegible] at a local company [illegible] needs and abilities of the student [illegible] particular [illegible]

[illegible] limitations, and [illegible] by a [illegible] self-study [illegible] because, [illegible]

[illegible] could provide the student [illegible]. The use of [illegible] and individual materials could [illegible]

[illegible]

of ideas, diagnostic post-tests, the remedial or enrichment activities of Phase II, and the opportunity to receive instruction at a level corresponding to the needs and abilities of the students for a particular topic should prove to be an effective instructional procedure.

Implications for Further Research

Although the evaluation of the Hardisty Project encompassed three studies, several studies concerning pupil achievement within the cognitive domain, as well as other factors involved in the Project could be made. In any study investigating pupil achievement, the testing instrument is of prime importance. One study which could be made could include the several pilot studies necessary for the construction of a valid and reliable testing instrument. The testing instrument could be designed to test more than the average level objectives common to both the individualized modes of instruction and the regular classroom instruction. Advanced objectives studied by the Advanced group could be included in the test. Enrichment objectives could also be tested. The testing instrument should consist of several subtests so that comparison of pupil achievement could be made on each of the various topics covered, and also at the various levels of the cognitive domain as classified by Bloom (1956). Further testing, as suggested, would provide more information

as to where the differences in pupil achievement existed.

Modifications could be made to the instructional mode developed. Further provision for learning styles could be made. It could be postulated that if the student had more learning styles to choose from, to meet his own particular needs, his achievement should improve. The inclusion of manipulative materials and audio-visual materials in the instructional mode would provide for two more learning styles.

An investigation could be made as to which student personality types can achieve better when receiving the individualized mode of instruction. It could also be determined how student attitude relates to achievement and to the mode of instruction used. Due to the large number of diagnostic tests given to students in the individualized mode of instruction, it might be expected that a change in test anxiety would be indicated by the students. Change in test anxiety could also be investigated.

The investigations performed in this study support the use of the individualized mode of instruction which was developed. Various other ways of providing for individual differences may be developed, and an infinite number of variables may influence the effectiveness of these programs with respect to cognitive achievement.

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A P P E N D I X A

GENERAL OBJECTIVES

APPENDIX A

GENERAL OBJECTIVES -- corresponding to the content area of rational numbers, rates and percent at the grade seven level issued by the Alberta Department of Education.

(1) Understanding fractions or rational numbers of arithmetic.

(2) The position and order of fractional numbers on the numberline.

(3) The ability to perform operations accurately on fractional numbers.

(4) Recognition and identification of number system properties in the fractional number system.

(5) Decimal numeral representation of fractional numbers and the expansion of the decimal numeration system.

(6) Computations using decimal numbers.

(7) Transformation of fractional numbers into decimal form; repeating and non-terminating decimals.

(8) Simplifications of expressions and the solution of problems involving fractions.

(9) Development of the concepts, common and unique properties of ratio and rate.

(10) The meaning of percent.

(11) Transformation of percent into decimal and fractional equivalents.

(12) Solutions of problems using and involving rates, ratio and percent.

A P P E N D I X B

CONTENT COVERED

APPENDIX B

CONTENT COVERED

Topics studied by students receiving the individualized mode of instruction:

Topic 1 -- Rational Numbers and Fractions

Section

1 Fractions represented by diagrams
2 Diagrams to represent fractions
3 Equivalent Fractions
4 Equivalent Fractions continued
5 Basic Fractions
6 Whole Numbers Represented by Fractions
7 Rational Numbers and the Number Line
8 Testing for Equivalence of Fractions
9 Ordering Rational Numbers
10 Density of Rational Numbers
11 Applications

Topic 2 -- Operations with Rational Numbers

Section

1 Operations of addition and subtraction with Rational Numbers
2 Mixed numerals and fractions
3 Addition and subtracting involving rational numbers named by mixed numbers
4 Solving conditions for equality with rational numbers
5 Applications
6 Rational Numbers and "of"
7 Operation of multiplication with rational numbers
8 Reciprocals
9 The operation of division with rational numbers
10 Conditions for equality involving products of rational numbers
11 Applications
12 Properties of operations with rational numbers

Topic 3 -- Decimals

Section

1 Decimal Place Value
2 Converting between fractions and decimals
3 Understanding addition and subtraction involving decimals

4 Solving conditions involving decimals and addition or subtraction
5 Applications
6 Rounding or approximating numbers named by decimals
7 Understanding multiplication involving decimals
8 Understanding division involving decimals
9 Solving conditions for equality involving multiplication and division of rational numbers named by decimals
10 Applications
11 Changing fractions to decimals
12 Changing decimals to basic fractions
13 Scientific or standard notation

Topic 4 -- Ratios, Proportion and Percent

Section

1 Ratios
2 Proportions
3 Solving Proportions
4 Situations Involving Proportions
5 Applications with Proportions
6 Comparisons Using Proportion
7 Percent
8 Percents, fractions and decimals
9 Applications of Percent - Part 1
10 Applications of Percent - Part 2 - Discount
11 Applications of Percent - Part 3 - Interest
12 Estimating Percent

A P P E N D I X   C

SAMPLE OF MATERIALS

Operating Procedures
Record Page
Flow Chart
Phase I
Phase II - Basic
Phase II - Intermediate
Phase II - Advanced
Challengers
Post Test I
Post Test II - Basic
Post Test II - Intermediate
Post Test II - Advanced

OPERATING PROCEDURES FOR INDIVIDUALIZATION OF GRADE VII MATHEMATICS

Prepared by Dr. R. S. Mortlock
University of Alberta

PHASE I

Materials:

(a) Flow Chart
- With the flow chart each student keeps a record of his progress through each topic
- Flow charts should be kept up to date and available for teacher reference
- A colored pencil can be used to trace the student's path

(b) Introduction
- Each topic has an introduction which briefly gives the student some ideas about what the topic covers, why it is important, how the ideas in the topic are used.

(c) Objectives
- These are on pink paper at the beginning of each section
- Each objective contains a statement of what the student is to be able to do, an example of what he is to be able to do, the criterion or standard of performance he must reach to achieve the objective, and a solution or answer (in a box) for the example
- There are two types of objectives. Basic (B) objectives (in regular type) and Intermediate (I) objectives (in italic type)
- The objectives are numbered for reference purposes

(d) Description
- This is on white paper for each section
- The description teaches the student what is to be learned for the section. It contains the mathematics for the section.
- The description may be in two types of type--regular and italic. The regular type covers the description for the B objectives. The italic type covers the description for the I objectives.
- Numbered questions are asked during the description. Answers to questions are given at the bottom of the page on which the questions occur. Whenever possible the student is asked to write an answer to the question (in his workbook, not in the description). Questions are of two types. Some precede the presentation of an idea and are designed to get the student to think about the idea before it is presented. Others follow the presentation of an idea and are designed to give the student immediate practice with the idea.
- Reference to objectives. It is expected that students will usually read and work through the presentation of a section before reading the objectives for the section. At the end of the description for

each section, students are directed to read the objectives which they are expected to achieve for the section (and the examples, criteria, and solutions). If for a certain objective, there is only a B objective then students are directed to read that objective and to be able to achieve it. If for a certain objective there is a B and an I objective, then students are directed to read the I objective and to be able to achieve it.

- Direction to Check Exercises. The last sentence in the description directs the student to the first Check Exercise for the section.

(e) First Check Exercise

- When the student has completed the description, read the objectives and their examples and feels able to do what the objectives require, he attempts first Check Exercise for the section. The Check Exercise is on the first yellow page for the section.
- The Check Exercise is to be answered in the student's workbook--not on the yellow pages.
- On completion of the Check Exercise, the student is directed to check his answer (answers are given at the very end of the topic). A statement tells him what he must have done to be successful on the Check Exercise.
- If successful, he goes on to the next section. If not successful he is directed to do some activities and exercises.

(f) Activities and Exercises

- These follow the first Check Exercise on the yellow pages. They are designed to give the students who were not successful on the first Check Exercise some more instruction in and practice with the ideas in the section.
- These exercises and activities are to be worked in the student's workbook, not on the yellow pages.
- Answers to all activities and exercises are given at the very end of the topic.
- On completing the activities and exercises, the student is directed to read the relevant objectives and then to do the second Check Exercise.

(g) Second Check Exercise

- These exercises are labelled with an (a).
- On completion of the second Check Exercise, the student checks his answer with the answer given at the end of the topic. If correct, he goes on to the next section. If incorrect, he is referred to a reference source (name of book, page and topic are given) or to a student helper or to his teacher for assistance since clearly help is needed. After help the student goes on to the next section.

(h) Summary and Vocabulary

- Following the last section of each topic, there is a summary for the topic which lists the main ideas developed in each section of the topic.

- After the summary there is a vocabulary list which defines the new words introduced in the topic. This vocabulary list should be brought to the student's attention early since at various stages they may wish to refer to the list to know what words mean.

(i) Review Exercises
- The last item for each Phase I of a topic is a set of review exercises which cover all objectives that the student is expected to achieve for the topic.
- These exercises together with the summary serve to bring all the ideas of the topic together and to review them before the first Post-test for the topic.

POST-TEST I

- There are 3 equivalent forms of this test available. These forms should be given randomly to the students. All items are keyed to their corresponding objectives.
- Post-Test I tests the students' achievement of the objectives at which instruction in Phase I has been aimed, i.e., I objectives or B objectives for which there are no I objectives
- Post-Test I should be graded according to the marking scale which indicates the performance required on each item to achieve the objective for the item.
- Post-Test I is scored by the number of objectives achieved.

RECORD PAGE

- There are separate record pages for each topic
- On each record page all the objectives for the topic are listed
- The boundary scores for the groups are indicated on the page
- From the student's total of objectives achieved on Post-Test I, his group is selected.
- The dash against each B or I objective means that this objective was tested on Post-Test I
- If the objective was achieved on Post-Test I a ✓ is placed on the dash.
- For students in the B group dashes need to be placed in the B column against B objectives for which there are also I objectives. If a student achieves an I objective in Phase I for which there is also a B objective the dash placed against the B objective is checked (✓). If the I objective was not achieved, then the student is expected to work on the B objective in Phase II. For students in this group, all I objectives are ignored in Phase II.
- Unchecked dashes in the B column for the B group, in the Box I columns for the I group and in the B, I, or A columns for the A group, indicate objectives to be worked on in Phase II.

PHASE II

Grouping for Phase II
-Depending on success on Post-Test I students are grouped in Phase II

Basic Group:	$\leq$ 50% of objectives achieved
Intermediate Group:	50-89% of objectives achieved
Advanced Group:	$\geq$ 90% of objectives achieved

- Some latitude may be allowed for students near the boundaries. They may be placed in the higher group than their score would indicate (i.e., they score just below the boundary) if they have previously been consistently in that group. This may depend on the sort of errors they made on Post-Test I.
- Students placed in the Basic Group for Phase II only work on unachieved B objectives in Phase II. If they achieved an I objective in Phase I than the corresponding B ojbective (if there is one) is assumed to have been achieved. If an I objective is not achieved on Post-Test I than the corresponding B objective (if there is one) is also considered to be not achieved.
- Students placed in the Intermediate Group for Phase II work on any unachieved B or I objectives from Phase I.
- Students placed in the Advanced Group for Phase II work on A objectives together with any unachieved B or I objectives from Phase I.

B Group

Materials

(a) Flow Chart
- There is a new flow chart at the beginning of the materials for the B group. It contains only B objectives. On it students mark their progress during Phase II. Achieved objectives are by-passed on the chart.
- By referring to his record page, the student circles on his flow chart the objectives he has to work to achieve in Phase II.

(b) Activities and Exercises
- Each student in the Basic Group gets a new set of activities and exercises for each section of the topic
- Activities and exercises are only done for objectives unachieved on Post-Test I
- Basic Group students are directed to:
  (1)read each unachieved objective and the appropriate section of the description
  (2)do the activities and exercises for any B objective they did not achieve on Post-Test I for each section of the topic. Their record page and flow chart tells them which objectives were not achieved. Answers to the activities and exercises are at the end of the Phase II materials.
  (3)re-read the B objective
  (4)do the Check Exercise for any unachieved B objectives for each section. Answers to Check Exercises are also given at the end of the Phase II materials.
  (5)go on to the next section for which they have an unachieved B objective if successful on the Check Exercise.

(6) ask their teacher for help if still not successful on the Check Exercise.
(7) do some more exercises after getting help from their teacher.
(8) go on to the next section after checking their answers to the exercises.

I Group

Materials

(a) Flow Chart
- Students in the I Group use their original flow chart on which they trace their progress in Phase II in a different color. Achieved objectives are bypassed.
- By referring to his record page, the student circles on his flow chart the objectives he has to work to achieve in Phase II.

(b) Exercises
- For the I group there is simply a set of exercises for the objectives in each section that are to be achieved. These objectives are the same as those to be achieved in Phase I.
- For any objectives they did not achieve on Post-Test I, students are directed to re-read the original description, to re-read the objectives, and then to work through the relevant exercises.
- Answers to the exercises are given at the end of the set of exercises
- If a student still cannot do the exercises, he is directed to get help from his teacher.

A Group

Materials

(a) Flow Chart
- Students in the A Group get a separate flow chart for the A objectives.

(b) Objectives and Description
- These students also get a set of A objectives and the description related to them.
- Where there are check exercises and activities required for the A objectives these are incorporated along with the objectives and description.

(c) B and I Level Phase II Materials
- Students in the A group are expected to use the Phase I material to work on any unachieved objectives from Phase I.

POST-TESTS II

- When students in the B Group have finished their work for Phase II, they are given Post-Test IIB which tests just B objectives.

- They only do the items for objectives not achieved on Post-Test I. Their record page tells them which items to do. Before doing the test they should circle the numbers in front of the items they have to do.
- When students in the I group have finished their work for Phase II, they are given Post-Tests IIB and II-I. On Post-Test IIB, they do just the items for B objectives not achieved on Post-Test I. On Post-Test II-I, they do just the items for I objectives not achieved on Post-Test I. Their record page tells them which items to do. Before doing the last they should circle the numbers in front of the items they have to do.
- When students in the A group have finished their work for Phase II, they are given Post-Tests IIB, II-I, and A. On Post-Tests IIB and II-I they do as for the I Group above. They do all items on Post-Test A.
- For all these post-tests, there are two equivalent forms which are given to students randomly.
- Post-Tests II are graded according to the marking scale which indicates the performance required on each item to achieve the objective for the item.

RECORD PAGE

- Phase II of the record page is completed from the items achieved on Post-Tests II.

PROBLEM SOLVING

- On completion of Post-Test II for each topic the student progresses to the problem solving activity. This is on white paper.
- This section includes some discussion of what is involved in problem solving and an example of a problem and its solution.
- The problems are arranged in three sections of increasing difficulty, and within each section the problems are in order of difficulty.
- The problems are non-routine and are not simply word problems.
- The students are told to try the problems starting from the beginning in the first problem solving activity. Later, more able students may be directed to start in the second or even third section of the problems because the first section or first and second section problems may be too easy for them.
- Answers are given to the problems but answers only, i.e., not solutions. Using these answers, the student checks whether he is correct or not but if incorrect does not get help on how to solve the problem since no solution is given.
- If a student tries a problem and gets the wrong answer he should attempt the problem again.
- If a student tries a problem in a section and cannot do it after a reasonable effort, he is directed to try the next problem in the same section.
- After trying all the problems in a section, the student is directed to try over again the ones he could not do at the first try, before going

on to the next section's problems.
- Students are told to not necessarily expect to be successful on all the problems nor to expect to try all the problems.
- Solutions to the problems are available to teachers who should collect, mark, and comment on the students efforts on the problems. A section for recording results on the problem solving activity is on the record page for each topic. On it the problems successfully completed should be recorded together with those attempted.

THE NEXT TOPIC

- After the problem solving activity, the student goes on to the next topic.

ENRICHMENT

- Students who complete all aspects of a topic very rapidly are given the opportunity to select from various enrichment activities. A date may be set for the earliest possible commencement of each new topic.

TOPIC 2 RECORD PAGE NAME ____________________

CLASS ____________________

OBJECTIVES	B	I	A
B 1.1			
I 1.1			
B 1.2			
I 1.2			
B 2.1			
I 2.1			
B 3.1			
I 3.1			
B 3.2			
I 3.2			
B 4.1			
I 4.1			
B 5.1			
I 5.1			
I 6.1			
B 7.1			
I 7.1			
B 8.1			
I 8.1			
B 9.1			
I 9.1			
I 9.2			
B 10.1			
I 10.1			
B 11.1			
I 11.1			
I 12.1			
I 12.2			
I 12.3			
A 1			
A 2			
A 3			
A 4			
A 5			
A 6			
PHASE I Sub Total			
Total			
Possible	0	17	
	17		

CHALLENGERS (Non-Routine Problem Solving)

Problem Number	Attempted This Problem	Success	
		Student Assessed	Teacher Assessed
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			

≤ 7 BASIC

8 to 13 INTERMEDIATE

≥ 14 ADVANCED

PHASE II GROUP ______

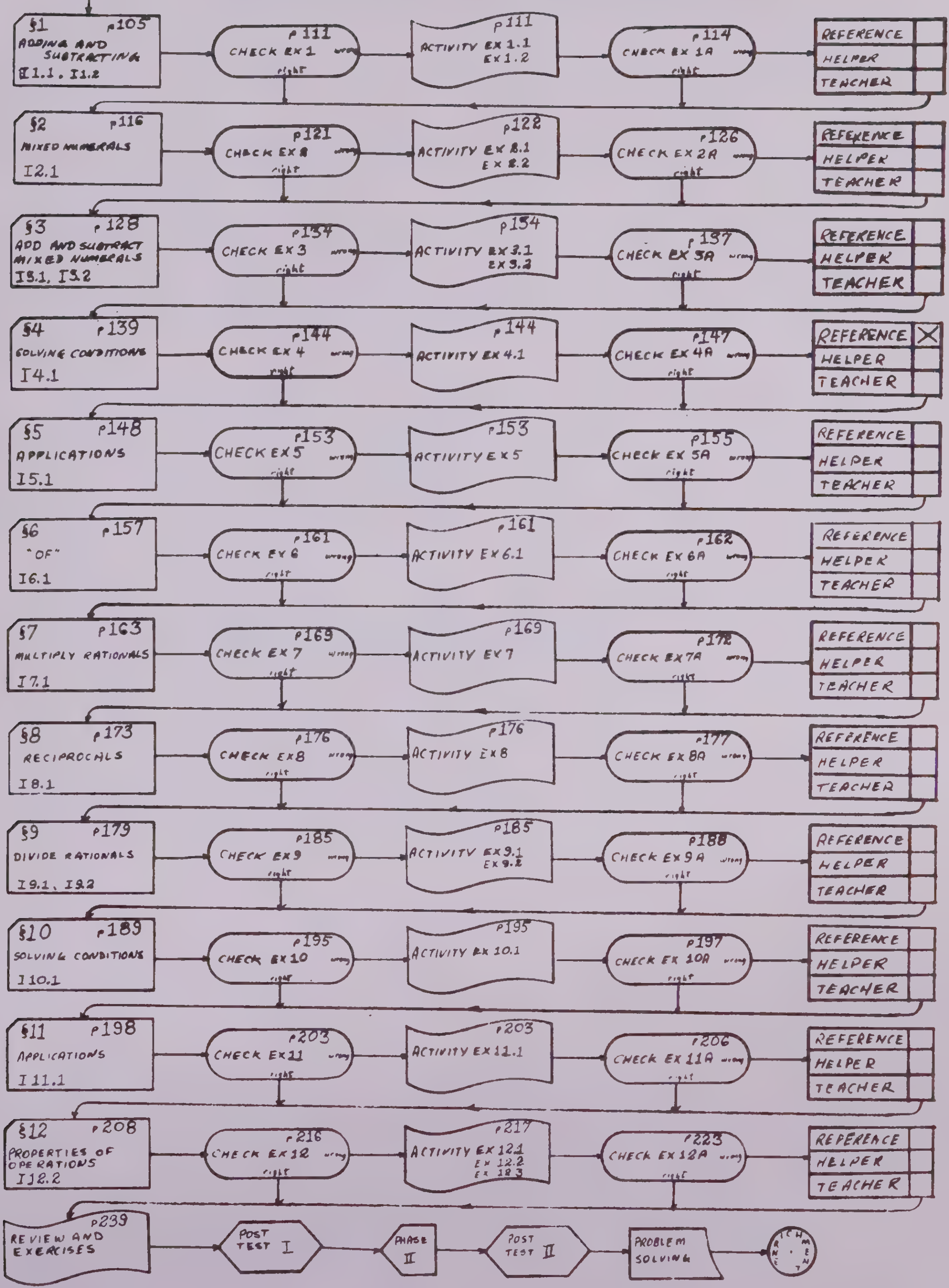
COMMENT: ______________________________

____

PHASE II B Group		PHASE I & II Total	
		Possible Total	12
PHASE II I Group		PHASE I & II Sub Total	
		Total	
PHASE II A Group		PHASE I & II Sub Total	

FLOWCHART

Phase I Topic II: Operations with Rational Numbers



TOPIC 2

PHASE I

OPERATIONS WITH RATIONAL NUMBERS

OPERATIONS WITH RATIONAL NUMBERS

INTRODUCTION

In almost every walk of life, at work or at home, people daily have to add, subtract, multiply or divide rational numbers; i.e. they have to use one or more of the four basic operations with rational numbers.

A girl may want to make a recipe for $\frac{2}{3}$ as many people as the directions on the packet indicate or may want to know the cost of $1\frac{3}{4}$ yards of material. A boy may want to know how much he would earn if he increased his paper round by $\frac{1}{3}$ or may want to know the combined length of a piece $1\frac{3}{4}$" long and one $2\frac{5}{8}$" long in a model he is planning.

Operations with rational numbers will also be used often in the work you will be doing in mathematics from now until you finish school. Since you will be using these ideas so much, it is important that you be able to use them accurately and quite quickly.

The operations that you will be studying in this topic are those listed above - addition, subtraction, multiplication and division. You have already met them before.

We will also be concerned with mixed numerals (eg. $4\frac{2}{5}$), with solving conditions involving rational numbers (eg. $n + \frac{2}{3} = \frac{3}{4}$), with reciprocals, (eg. $\frac{2}{3}$ and $\frac{3}{2}$), with properties of rational numbers (eg. commutative property) and with applying rational numbers to answer questions about everyday situations.

105

OBJECTIVE B1.1

To find the sum of two or more rational numbers named by fractions.

Example

Find the sums and write each as a basic fraction.

A. $\frac{3}{4} + \frac{5}{8} + \frac{1}{4}$

B. $\begin{array}{r} \frac{1}{3} \\ \frac{2}{5} \\ + \frac{5}{3} \\ \hline \end{array}$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A. $\frac{3}{4} + \frac{5}{8} + \frac{1}{4} = \frac{6}{8} + \frac{5}{8} + \frac{2}{8}$

$= \frac{13}{8}$

B. $\begin{array}{rcl} \frac{1}{3} & = & \frac{5}{15} \\ \frac{2}{5} & = & \frac{6}{15} \\ + \frac{5}{3} & = & \frac{25}{15} \\ \hline & & \frac{36}{15} = \frac{36 \div 3}{15 \div 3} = \frac{12}{5} \end{array}$

OBJECTIVE I1.1

To find the sum of two or more rational numbers named by fractions.

Example

Find the sums and write each as a basic fraction.

A. $\frac{1}{5} + \frac{1}{4} + \frac{5}{6}$

B. $\begin{array}{r} \frac{1}{9} \\ \frac{7}{6} \\ + \frac{1}{4} \\ \hline \end{array}$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A. $\frac{1}{5} + \frac{1}{4} + \frac{5}{6} = \frac{12}{60} + \frac{15}{60} + \frac{50}{60}$

$= \frac{77}{60}$

B. $\begin{array}{rcl} \frac{1}{9} & = & \frac{4}{36} \\ \frac{7}{6} & = & \frac{42}{36} \\ + \frac{1}{4} & = & \frac{9}{36} \\ \hline & & 55 \end{array}$

OBJECTIVE B1.2

To find the difference of two rational numbers named by fractions.

Example

Find the differences and write each as a basic fraction.

A. $\frac{7}{3} - \frac{5}{4}$

B. $\begin{array}{r} \frac{8}{9} \\ -\frac{1}{3} \\ \hline \end{array}$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A. $\frac{7}{3} - \frac{5}{4} = \frac{28}{12} - \frac{15}{12}$
$= \frac{13}{12}$

B. $\begin{array}{rcr} \frac{8}{9} & = & \frac{8}{9} \\ -\frac{1}{3} & = & \frac{3}{9} \\ \hline & & \frac{5}{9} \end{array}$

OBJECTIVE I1.2

To find the difference of two rational numbers named by fractions.

Example

Find the differences and write each as a basic fraction.

A. $\frac{7}{8} - \frac{1}{5}$

B. $\begin{array}{r} \frac{17}{5} \\ -\frac{5}{6} \\ \hline \end{array}$

Criterion: Method correct in each example and no more than one error in computation.

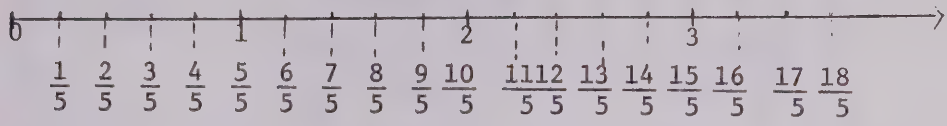
SOLUTION

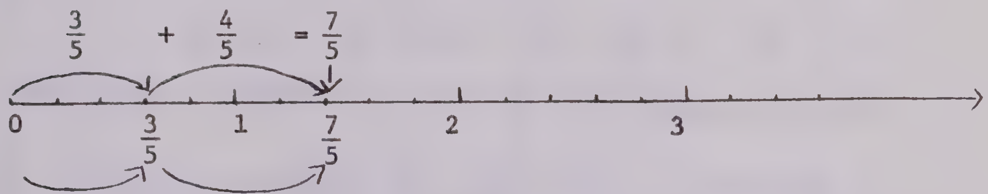
A. $\frac{7}{8} - \frac{1}{5} = \frac{35}{40} - \frac{8}{40}$
$= \frac{27}{40}$

B. $\begin{array}{rcr} \frac{17}{5} & = & \frac{102}{30} \\ -\frac{5}{6} & = & \frac{25}{30} \\ \hline & & \frac{77}{30} \end{array}$

SECTION 1. Operations of addition and subtraction with rational numbers

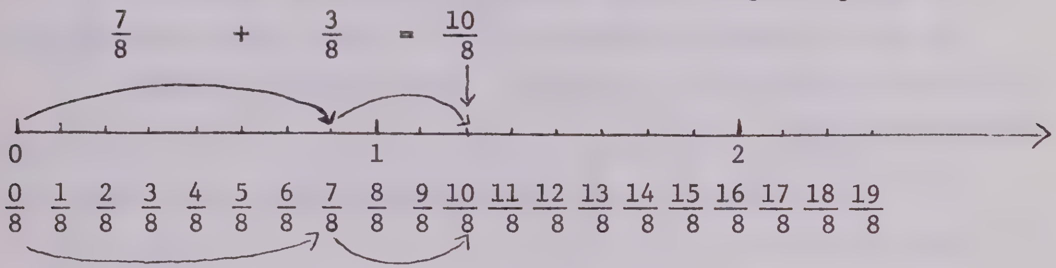
Suppose we want to add $\frac{3}{5}$ and $\frac{4}{5}$. We can show this on a numberline subdivided into fifths.





3 fifths + 4 fifths = 7 fifths

Here is another example, this time with eights. $\frac{7}{8} + \frac{3}{8}$



7 eights + 3 eights = 10 eights

The sum is $\frac{10}{8}$. Since answers are usually given as basic fractions we reduce $\frac{10}{8}$ to its basic fraction.

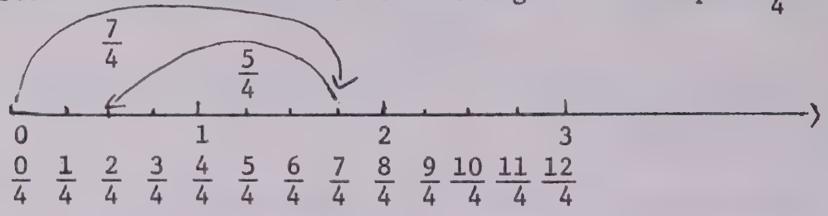
$$\frac{10}{8} = \frac{10 \div 2}{8 \div 2} = \frac{5}{4}$$

1. Use the numberlines above (if necessary) to find

   (1) $\frac{4}{5} + \frac{6}{5}$ (2) $\frac{3}{8} + \frac{9}{8}$

Answers: 1. (1) $\frac{10}{5} = \frac{2}{1} = 2$ (2) $\frac{12}{8} = \frac{3}{2}$

To subtract we do the same sort of thing. For example $\frac{7}{4} - \frac{5}{4}$



7 fourths − 5 fourths = 2 fourths

$$\frac{7}{4} - \frac{5}{4} = \frac{2}{4} \qquad = \frac{2 \div 2}{4 \div 4} = \frac{1}{2}$$

2. Write answers to the following in your work book.

(1) $\frac{7}{8} - \frac{3}{8}$ (2) $\frac{11}{16} - \frac{5}{16}$ (3) $\frac{2}{3} + \frac{4}{3}$ (4) $\frac{7}{4} - \frac{7}{4}$

3. Write the answer to $\frac{7}{4} + \frac{1}{2}$

In the examples we have done, the denominators have been the same in both fractions.

What about $\frac{5}{8} + \frac{1}{2}$?

In order to add them the denominators must be the same.

We can use common denominators and the least common denominator (L.C.D.) is usually most convenient. Here, the least common denominator is 8.

$\frac{5}{8} + \frac{1}{2}$ becomes $\frac{5}{8} + \frac{4}{8}$ and this sum is $\frac{9}{8}$.

4. Write the answer to $\frac{2}{3} + \frac{3}{4}$

Here is another example $\frac{2}{5} + \frac{5}{4}$

The L.C.D. is 20

$$\frac{2}{5} = \frac{2 \times 4}{5 \times 4} = \frac{8}{20} \quad \text{and} \quad \frac{5}{4} = \frac{5 \times 5}{4 \times 5} = \frac{25}{20}$$

Thus $\frac{2}{5} + \frac{5}{4} = \frac{8}{20} + \frac{25}{20}$

$= \frac{33}{20}$ (and this is a basic fraction)

Answers: 2: (1) $\frac{4}{8} = \frac{1}{2}$ (2) $\frac{6}{16} = \frac{3}{8}$ (3) $\frac{6}{3} = \frac{2}{1}$ or 2 (4) $\frac{0}{4}$ or 0

3: $\frac{7}{4} + \frac{1}{2} = \frac{7}{4} + \frac{1 \times 2}{2 \times 2} = \frac{7}{4} + \frac{2}{4} = \frac{9}{4}$ 4: $\frac{2}{3} + \frac{3}{4} = \frac{2 \times 4}{3 \times 4} + \frac{3 \times 3}{4 \times 3} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}$

Yet another example:

$\frac{5}{6} + \frac{9}{10}$ — Find the L.C.D. It is 30.

$= \frac{5 \times 5}{6 \times 5} + \frac{9 \times 3}{10 \times 3}$ — Find equivalent fractions with this denominator.

$= \frac{25}{30} + \frac{27}{30}$ — Add the numerators.

$= \frac{52}{30}$ — Is the answer a basic fraction?

$= \frac{52 \div 2}{30 \div 2}$ — If not, reduce it to a basic fraction.

$= \frac{26}{15}$

Now a subtraction example. The steps are similar.

$\frac{9}{8} - \frac{5}{6}$ — Find the L.C.D. It is 24

$= \frac{9 \times 3}{8 \times 3} - \frac{5 \times 4}{6 \times 4}$ — Find equivalent fractions with this denominator. Find the difference of the numerators.

$= \frac{27}{24} - \frac{20}{24}$ — The answer is a basic fraction. We are finished.

$= \frac{7}{24}$

Now look at this. Suppose in the last example we had used 48 as the common denominator instead of 24. The steps would be:

$\frac{9}{8} - \frac{5}{6}$

$= \frac{9 \times 6}{8 \times 6} - \frac{5 \times 8}{6 \times 8}$

$= \frac{54}{48} - \frac{40}{48}$

$= \frac{14}{48}$ ← The result here is not a basic fraction. However, reducing it we get the same as above.

$= \frac{14 \div 2}{48 \div 2}$

$= \frac{7}{24}$

We can use any common denominator when adding or subtracting rational numbers. However, the least common denominator usually gives the result with the least amount of work.

5. Find $\frac{2}{3} + \frac{3}{5} + \frac{7}{4}$

To find the sum of three or more rational numbers, you find the L.C.D. of <u>all</u> the denominators, find equivalent fractions with this denominator and then add the numerators.

For example:

Find

$$\begin{array}{rcccc} & \frac{5}{6} & = \frac{5 \times 6}{6 \times 6} & = & \frac{30}{36} \\ & \frac{5}{4} & = \frac{5 \times 9}{4 \times 9} & = & \frac{45}{36} \\ + & \frac{5}{9} & = \frac{5 \times 4}{9 \times 4} & = & \frac{20}{36} \\ \hline & & & & \frac{95}{36} \end{array}$$

The L.C.D. of 6, 4, and 9 was 36.

Now read OBJECTIVES I1.1 and I1.2 and their examples. These tell you what you are expected to be able to do for this section. When ready turn to and do CHECK EXERCISES 1.1 and 1.2.

Answers: 5: $\frac{2}{3} + \frac{3}{5} + \frac{7}{4} = \frac{2 \times 20}{3 \times 20} + \frac{3 \times 12}{5 \times 12} + \frac{7 \times 15}{4 \times 15} =$

$\frac{40}{60} + \frac{36}{60} + \frac{105}{60} = \frac{181}{60}$

CHECK EXERCISE 1

I1.1 Add the following and write each sum as a basic fraction.

a) $\frac{6}{25} + \frac{3}{10} + \frac{4}{5}$

b) $\frac{5}{8} + \frac{11}{16} + \frac{2}{3}$

c) $\frac{3}{4} + \frac{11}{15} + \frac{7}{12}$

d) $\begin{array}{r} \frac{9}{10} \\ \frac{3}{4} \\ + \frac{2}{5} \\ \hline \end{array}$

e) $\begin{array}{r} \frac{9}{10} \\ \frac{11}{12} \\ + \frac{3}{4} \\ \hline \end{array}$

I1.2 Find the differences and write each as a basic fraction.

a) $\frac{2}{3} - \frac{1}{6}$

b) $\frac{4}{5} - \frac{1}{2}$

c) $\frac{4}{9} - \frac{1}{6}$

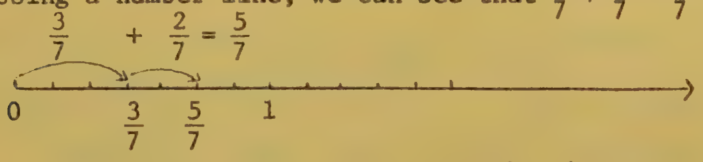
d) $\frac{7}{8} - \frac{1}{5}$

e) $\frac{10}{6} - \frac{3}{4}$

- Check your answers with those given at the end of the topic.
- If you are not certain how to add rational numbers, or if you had more than one part incorrect in CHECK EXERCISE 1.1; read section 1 carefully and do activity exercises 1.1.
- If you are not certain how to subtract rational numbers, or if you had more than one part incorrect in CHECK EXERCISE 1.2; read section 1 carefully and do activity exercises 1.2.
- Otherwise, go on to section 2.

Activity Exercises 1.1

1. Using a number line, we can see that $\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$



a) Write the fraction that equals $\frac{2}{7} + \frac{4}{7}$.

b) Add the following. Use a number line if you wish, but try to do them without.

i) $\frac{4}{9} + \frac{3}{9}$ ii) $\frac{5}{4} + \frac{2}{4}$ iii) $\frac{3}{10} + \frac{8}{10}$ iv) $\frac{1}{5} + \frac{3}{5}$

$\frac{3}{8} + \frac{1}{4}$ cannot be added as easily because they have different denominators.

But $\frac{1}{4} = \frac{1 \times 2}{4 \times 2} = \frac{2}{8}$. Now we have $\frac{3}{8} + \frac{2}{8} = \frac{5}{8}$.

Study the following examples:

a) $\frac{3}{4} + \frac{2}{5}$

$= \frac{3 \times 5}{4 \times 5} + \frac{2 \times 4}{5 \times 4}$ L.C.D. is 20

$= \frac{15}{20} + \frac{9}{20}$

$= \frac{23}{20}$

b) $\frac{1}{4} + \frac{5}{6}$

$= \frac{1 \times 3}{4 \times 3} + \frac{5 \times 2}{6 \times 2}$ L.C.D. is 12

$= \frac{3}{12} + \frac{10}{12}$

$= \frac{13}{12}$

Note: We must have common denominators in order to add rational numbers. The least common denominator is usually most convenient.

c) Add the following. Be sure you have a common denominator.

i) $\frac{1}{2} + \frac{3}{4}$

ii) $\frac{2}{3} + \frac{1}{6}$

iii) $\frac{8}{5} + \frac{4}{3}$

iv) $\frac{4}{7} + \frac{5}{9}$

v) $\frac{11}{10} + \frac{11}{7}$

Adding $\frac{3}{10} + \frac{3}{10}$ we get $\frac{6}{10}$. But $\frac{6}{10}$ can be reduced to a basic fraction of $\frac{3}{5}$. $\left(\frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}\right)$.

Note: We always reduce a fraction to its basic fraction before considering the question finished.

d) Add the following. Be sure they are reduced to basic fractions when necessary.

i) $\frac{5}{12} + \frac{1}{4}$

ii) $\frac{1}{15} + \frac{5}{6}$

iii) $\frac{3}{10} + \frac{7}{8}$

Adding three or more rational numbers is done just as for adding two rational numbers. Be sure they <u>ALL</u> are expressed with common denominators and the final answers are BASIC FRACTIONS.

$\frac{2}{3} + \frac{4}{5} + \frac{5}{6}$

$= \frac{20}{30} + \frac{24}{30} + \frac{25}{30}$ - common denominators (L.C.M. is 30)

$= \frac{69}{30}$

$= \frac{23}{10}$ - basic fraction

We can also use column form when adding 2 or more fractions.

$$\begin{array}{rcrl} \frac{3}{8} & = & \frac{9}{24} & \\ \frac{5}{6} & & \frac{20}{24} & \text{L.C.D. is 24} \\ + \ \frac{2}{3} & & + \ \frac{16}{24} & \\ \hline & & \frac{45}{24} & \\ & & = \frac{15}{8} & \text{basic fraction} \end{array}$$

e) Add the following and express the sums as basic fractions.

i) $\frac{5}{6} + \frac{1}{2} + \frac{2}{3}$

ii) $\frac{1}{2} + \frac{3}{8} + \frac{1}{4}$

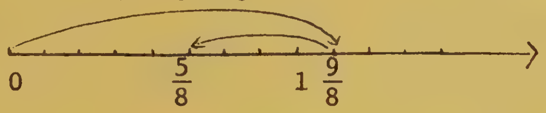
iii) $\frac{3}{8} + \frac{5}{12} + \frac{8}{15}$

iv)
$$\begin{array}{r} \frac{2}{3} \\ \frac{2}{5} \\ + \ \frac{7}{15} \\ \hline \end{array}$$

v)
$$\begin{array}{r} \frac{2}{3} \\ \frac{3}{4} \\ + \ \frac{5}{6} \\ \hline \end{array}$$

Activity Exercises 1.2

We can also use the number line to find the difference between two rational numbers. $\frac{9}{8} - \frac{4}{8} = \frac{5}{8}$



a) Subtract the following. Use a number line if necessary, but try to do it without.

i) $\frac{5}{9} - \frac{4}{9}$ ii) $\frac{12}{5} - \frac{8}{5}$ iii) $\frac{7}{8} - \frac{5}{8}$

In order to subtract $\frac{4}{5}$ from $\frac{9}{10}$ we must express both fractions with a common denominator.

$$\frac{9}{10} - \frac{4}{5}$$
$$= \frac{9}{10} - \frac{8}{10} \qquad \left(\frac{4}{5} = \frac{4 \times 2}{5 \times 2} = \frac{8}{10} \right)$$
$$= \frac{1}{10}$$

b) Subtract the following. Use the least common denominator.

i) $\frac{9}{5} - \frac{2}{15}$ ii) $\frac{5}{8} - \frac{2}{6}$ iii) $\frac{5}{7} - \frac{2}{5}$

In the following example we find we get an answer of $\frac{2}{12}$.

$$\frac{7}{12} - \frac{5}{12} = \frac{2}{12}$$

But, we do not leave the answer as $\frac{2}{12}$. We reduce it to its BASIC FRACTION.

$$\frac{2}{12} = \frac{2 \div 2}{12 \div 2} = \frac{1}{6}$$

Thus $\frac{7}{12} - \frac{5}{12} = \frac{2}{12}$

$$= \frac{1}{6}$$

We can also subtract in column form.

$$\begin{array}{rcr} \frac{4}{5} & = & \frac{4}{30} \\ -\ \frac{1}{10} & & -\ \frac{3}{30} \\ \hline & & \frac{1}{30} \end{array}$$

NOTE: ALWAYS EXPRESS THE ANSWER AS A BASIC FRACTION.

c) Find the differences and express each as a basic fraction.

i) $\frac{7}{16} - \frac{1}{4}$

ii) $\frac{3}{10} - \frac{1}{20}$

iii) $\frac{9}{20} - \frac{1}{5}$

iv) $\begin{array}{r} \frac{3}{4} \\ -\ \frac{2}{3} \\ \hline \end{array}$

v) $\begin{array}{r} \frac{9}{5} \\ -\ \frac{4}{7} \\ \hline \end{array}$

- Check your answers with those given at the end of the topic.
- Read objective I1.1 and I2.1.
- If you feel you are ready, do CHECK EXERCISE 1A.

CHECK EXERCISE 1A

I1.1 Add the following and write each sum as a basic fraction.

a) $\frac{1}{4} + \frac{2}{9} + \frac{5}{12}$

b) $\frac{1}{4} + \frac{4}{25} + \frac{3}{10}$

c) $\frac{3}{14} + \frac{4}{21} + \frac{5}{28}$

d) $\begin{array}{r} \frac{1}{6} \\ \frac{2}{3} \\ +\ \frac{0}{5} \\ \hline \end{array}$

e) $\begin{array}{r} \frac{3}{8} \\ \frac{1}{6} \\ +\ \frac{5}{12} \\ \hline \end{array}$

I1.2 Find the differences and write each as a basic fraction.

a) $\frac{7}{12} - \frac{1}{4}$

b) $\frac{7}{15} - \frac{1}{6}$

c) $\frac{5}{8} - \frac{1}{3}$

d) $\frac{11}{16} - \frac{7}{12}$

e) $\frac{5}{6} - \frac{8}{15}$

- Check your answers with those given at the end of the topic.
- If you are not sure how to add or subtract rational numbers, or if you had more than one error in each CHECK EXERCISE, check Modern School Mathematics pp. 336-340 or ask a student helper or ask your teacher.
- Otherwise, go on to section 2.

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OBJECTIVE B2.1

(1) To write a fraction with numerator greater than denominator as a mixed numeral.

(2) To write a mixed numeral as a fraction.

Example

(1) Write as a mixed numeral $\frac{35}{8}$

(2) Write as a fraction $7\frac{2}{3}$

SOLUTION

$$\frac{35}{8} = \frac{32}{8} + \frac{3}{8} = 4\frac{3}{8}$$

$$7\frac{2}{3} = \frac{21}{3} + \frac{2}{3} = \frac{23}{3}$$

Criterion: Correct mixed numeral and fraction.

OBJECTIVE I2.2

a) To use fractions to justify that a given fraction has a particular mixed numeral.

b) To use fractions to justify that a given mixed numeral has a particular fraction.

Example

a) Use fractions to justify that the mixed numeral for $\frac{22}{5}$ *is* $4\frac{2}{5}$.

b) Use fractions to justify that the fraction for $5\frac{2}{3}$ *is* $\frac{17}{3}$.

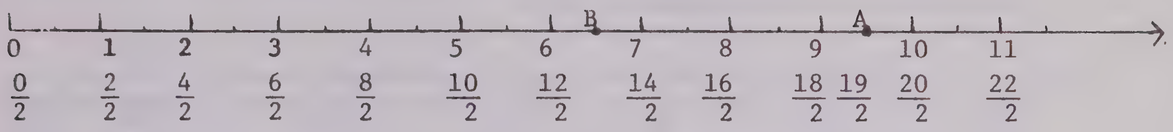
Criterion: Correct justifications.

SOLUTION

a) $$\frac{22}{5} = \frac{20}{5} + \frac{2}{5} = \frac{20 \div 5}{5 \div 5} + \frac{2}{5} = \frac{4}{1} + \frac{2}{5} = 4 + \frac{2}{5} = 4\frac{2}{5}$$

b) $$5\frac{2}{3} = 5 + \frac{2}{3} = \frac{5}{1} + \frac{2}{3} = \frac{5 \times 3}{1 \times 3} + \frac{2}{3} = \frac{15}{3} + \frac{2}{3} = \frac{17}{3}$$

Section 2. Mixed numerals and fractions



1. Write two ways of naming the rational number associated with point B on the number line.

The rational number associated with point A on the number line above may be named in two ways; $\frac{19}{2}$ or $9\frac{1}{2}$. The second of the names is the one often used in everyday situations; e.g. $9\frac{1}{2}$ inches, $9\frac{1}{2}$ years, etc.

On the number line subdivided into halves, $9\frac{1}{2}$ means 9 whole divisions plus one of the halves subdivisions. This is the same as 19 subdivisions, each one half.

$9\frac{1}{2}$ is called a mixed numeral because it contains both a whole number numeral and a fraction.

$9\frac{1}{2}$ ← fraction	$9\frac{1}{2}$	$9\frac{1}{2} = 9 + \frac{1}{2}$
whole number numeral	mixed numeral	meaning

When we get to applications of rational numbers to answer questions about everyday situations we will find that mixed numerals are often used. We will also need to be able to convert between mixed numerals and fractions so let's see how this is done.

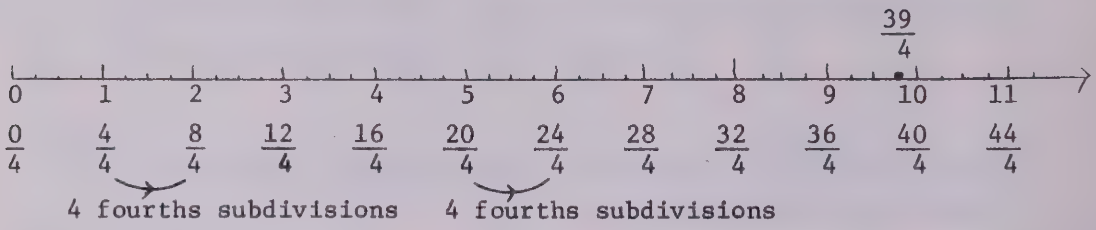
Fractions to mixed numerals

Of course, it is only when the numerator of the fraction is greater than the denominator that this can be done.

2. Write the mixed numeral for $\frac{17}{4}$.

Answers: 1: $\frac{13}{2}$ or $6\frac{1}{2}$ 2: 16 fourths are 4. Thus $\frac{17}{4} = 4\frac{1}{4}$.

Consider the fraction $\frac{39}{4}$. In terms of a number line subdivided into fourths, the point corresponding to $\frac{39}{4}$ is 39 fourths subdivisions from the point corresponding to 0.



It takes 4 fourths subdivisions to make each whole division. How many whole subdivisions are included in 39 fourths subdivisions? We find out by dividing 39 by 4.

$$\begin{array}{r} 9 \\ 4\overline{)39} \\ \underline{36} \\ 3 \end{array}$$

39 fourths subdivisions is 9 whole divisions and 3 fourths subdivisions.

i.e. $\frac{39}{4} = 9\frac{3}{4}$

This can also be justified, using fractions, as follows:

$$\frac{39}{4} = \frac{36}{4} + \frac{3}{4} = \frac{36 \div 4}{4 \div 4} + \frac{3}{4} = \frac{9}{1} + \frac{3}{4} = 9 + \frac{3}{4} = 9\frac{3}{4}$$

Let's try $\frac{104}{16}$ and obtain the mixed numeral for it.

First divide, to find the number of wholes.

$$\begin{array}{r} 6 \\ 16\overline{)104} \\ \underline{96} \\ 8 \end{array} \qquad 6\frac{8}{16}$$

But $\frac{8}{16}$ can be reduced to the basic fraction $\frac{1}{2}$

Thus $\frac{104}{16} = 6\frac{8}{16} = 6\frac{1}{2}$

Justification using fractions

$$\begin{aligned} \frac{104}{16} &= \frac{96}{16} + \frac{8}{16} \\ &= \frac{96 \div 16}{16 \div 16} + \frac{8 \div 8}{16 \div 8} \\ &= \frac{6}{1} + \frac{1}{2} \\ &= 6 + \frac{1}{2} \\ &= 6\frac{1}{2} \end{aligned}$$

3. Write $\frac{100}{8}$ as a mixed numeral.

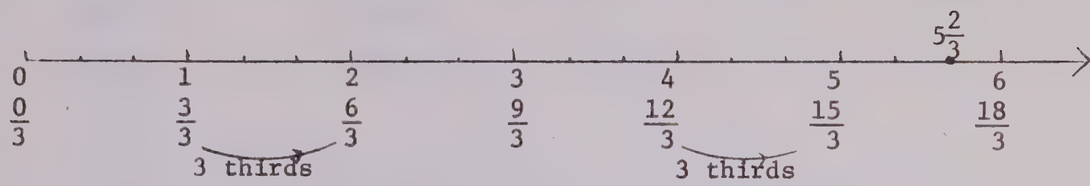
<u>Mixed numerals to fractions</u>

This can be done for every mixed numeral.

Answers: 3: $8\overline{)100}$ 12 remainder 4 $\qquad \frac{100}{8} = 12\frac{4}{8} = 12\frac{1}{2}$

4. Write the fraction for $4\frac{2}{3}$.

Consider the mixed numeral $5\frac{2}{3}$.



Each whole division has 3 thirds subdivisions.

5 whole divisions have 5 x 3 = 15 thirds subdivisions.

$5\frac{2}{3}$ has 15 thirds subdivisions plus 2 more thirds subdivisions;
i.e. 17 thirds subdivisions.

Thus $5\frac{2}{3} = \frac{17}{3}$

This can also be justified using fractions as follows:

$$5\frac{2}{3} = 5 + \frac{2}{3} = \frac{5}{1} + \frac{2}{3} = \frac{5 \times 3}{3 \times 3} + \frac{2}{3} = \frac{15}{3} + \frac{2}{3} = \frac{17}{3}$$

Let's try $4\frac{7}{12}$ and obtain the fraction for it.

First multiply to find the number of twelfths in 4

4 is 4 x 12 = 48 twelfths

$4\frac{7}{12}$ is (48 + 7) twelfths

$4\ \frac{7}{12} = \frac{55}{12}$

Justification using fractions

$$\begin{aligned} 4\ \frac{7}{12} &= 4 + \frac{7}{12} \\ &= \frac{4}{1} + \frac{7}{12} \\ &= \frac{4 \times 12}{1 \times 12} + \frac{7}{12} \\ &= \frac{48}{12} + \frac{7}{12} \\ &= \frac{55}{12} \end{aligned}$$

5. Write the mixed numeral for $9\frac{8}{9}$.

For a number to be correctly named by a mixed numeral, the fraction part must have numerator less than denominator.

$3\frac{5}{4}$ is not written correctly. It should be $4\frac{1}{4}$.

Answers: 4: 4 is 12 thirds. Thus $4\frac{2}{3}$ is $\frac{14}{3}$.

5: 9 is 9 x 9 = 81 ninths. $9\frac{8}{9} = \frac{89}{9}$

Now read OBJECTIVES B2.1 and I2.2 and their examples. These tell you what you are expected to be able to do for this section. When ready, turn to and do CHECK EXERCISES 2.1 and 2.2.

CHECK EXERCISE 2

B2.1 i) Write the following fractions as mixed numerals:

a) $\frac{37}{5}$ d) $\frac{13}{7}$

b) $\frac{9}{4}$ e) $\frac{16}{9}$

c) $\frac{11}{3}$

ii) Write the following mixed numerals as fractions:

a) $1\frac{3}{4}$ d) $4\frac{4}{9}$

b) $2\frac{5}{7}$ e) $5\frac{3}{8}$

c) $1\frac{8}{11}$

I2.2 i) Use fractions to justify that the mixed numeral for:

a) $\frac{15}{4}$ is $3\frac{3}{4}$

b) $\frac{22}{7}$ is $3\frac{1}{7}$

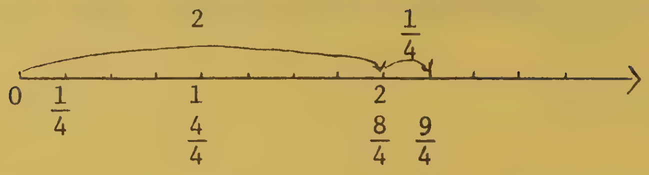
ii) Use fractions to justify that the fraction for:

a) $3\frac{3}{5}$ is $\frac{18}{5}$

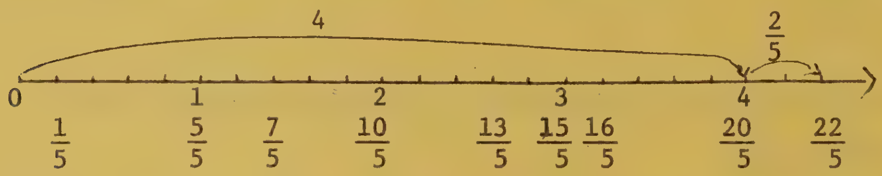
b) $2\frac{1}{4}$ is $\frac{9}{4}$

- Check your answers with those given at the end of the topic.
  You were successful on CHECK EXERCISE B2.1 if you had at least 4 of the parts correct in both 2.1 (i) and 2.1 (ii).
  You were successful on CHECK EXERCISE I2.2 if you had 3 of the 4 parts correct and all steps were shown.
- If you are not sure how to do either of the above CHECK EXERCISES or if you were unsuccessful on CHECK EXERCISE B2.1 read section 2 and do activity exercise 2.1.
  If you were unsuccessful on CHECK EXERCISE I2.2 read section 2 and do activity exercise 2.2.
- Otherwise, go on to section 3.

Activity Exercise 2.1



From the number line, we can see that $\frac{9}{4}$ means 2 whole units on the number line plus one-quarter of the next unit. This could be written as $2\frac{1}{4}$, which means $2 + \frac{1}{4}$. $2\frac{1}{4}$ is said to be a <u>mixed numeral</u> as it contains both a whole number and a fraction.



On the number line above we see that $\frac{22}{5}$ has four whole units divided into fifths plus $\frac{2}{5}$ of the next unit. ie. $\frac{22}{5} = 4\frac{2}{5}$

1. Using the above number line, write mixed numerals for:

   a) $\frac{7}{5}$ b) $\frac{13}{5}$ c) $\frac{16}{5}$

   Of course we do not want to always draw a number line to determine the mixed numeral. We can use the following method which involves division.

   We can find the mixed numeral for $\frac{22}{5}$ by dividing 22 by 5.

$$\begin{array}{r} 4 \\ 5\,\overline{)\,22} \\ \underline{20} \\ 2 \end{array} \quad 4\frac{2}{5}$$

   Four whole units and $\frac{2}{5}$ of the next unit.

   Similarly $\frac{26}{5} = 5\frac{1}{5}$ because

$$\begin{array}{r} 5 \\ 5\,\overline{)\,26} \\ \underline{25} \\ 1 \end{array} \quad \text{i.e. } 5\frac{1}{5}$$

2. Using the division method write mixed numerals for:

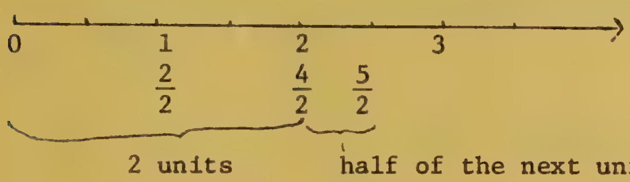
a) $\frac{15}{4}$ d) $\frac{31}{6}$

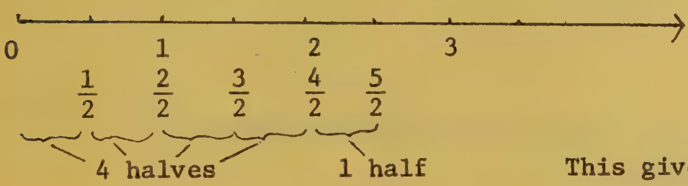
b) $\frac{17}{3}$ e) $\frac{24}{7}$

c) $\frac{21}{8}$

Given the mixed numeral we also want to be able to express it as a fraction.

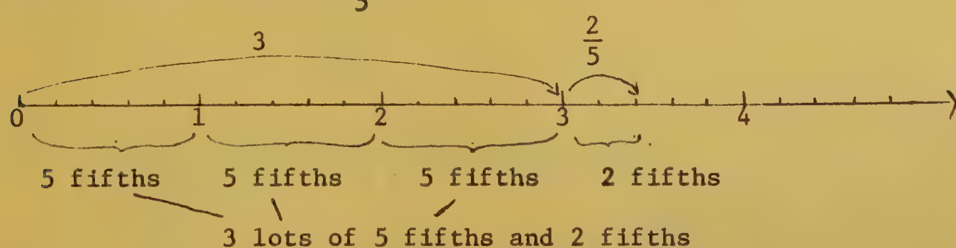
$2\frac{1}{2}$ means $2 + \frac{1}{2}$ or 2 units plus $\frac{1}{2}$ of the next unit.





This gives 5 halves or $\frac{5}{2}$.

Study the following example in which we write a fraction for the mixed numeral $3\frac{2}{5}$.



i.e. (3 x 5) fifths + 2 fifths

i.e. $\frac{15}{5} + \frac{2}{5}$

i.e. $\frac{17}{5}$

3. Write a fraction for $2\frac{3}{4}$.

Again we want to be able to write fractions equivalent to mixed numerals without having to draw a number line. Looking at the mixed numeral $3\frac{2}{5}$ we see that it is made up of 3 and $\frac{2}{5}$. To find the number of fifths in 3 we multiply 3 by 5. This gives us 15 fifths in the 3 plus 2 more fifths in the $\frac{2}{5}$. Thus we have 15 + 2 fifths, i.e. 17 fifths or $\frac{17}{5}$.

To do this quivkly we can think $3\frac{2}{5}$ = [(5 x 3) + 2] fifths

= [15 + 2] fifths

$= \frac{17}{5}$

$4\frac{2}{3}$ = [(3 x 4) + 2] thirds

= 12 + 2 thirds

$= \frac{14}{3}$

4. Write fractions for the following mixed numerals:

a) $5\frac{2}{5}$ d) $5\frac{1}{4}$

b) $3\frac{2}{3}$ e) $2\frac{7}{9}$

c) $1\frac{4}{7}$

Activity Exercise 2.2

We can use fractions to justify that a given fraction has a particular mixed numeral.

For example:

$$\begin{aligned} \frac{15}{4} &= \frac{12}{4} + \frac{3}{4} \\ &= \frac{12 \div 4}{4 \div 4} + \frac{3}{4} \\ &= \frac{3}{1} + \frac{3}{4} \\ &= 3 + \frac{3}{4} \\ &= 3\frac{3}{4} \end{aligned}$$

all these steps are necessary for the justification

1. Write out and complete the next example.

$$\frac{23}{5} = \frac{\square}{5} + \frac{\square}{5}$$
$$= \frac{20 \div \square}{5 \div \square} + \frac{3}{5}$$
$$= \frac{\square}{\square} + \frac{3}{5}$$
$$= \square + \square$$
$$= \square$$

(note: we want to form two fractions such that one may be simplified to a whole number <u>and</u> the other has numerator less than denominator).

2. Justify that each of the following fractions is the given mixed numeral:

a) $\frac{37}{6}$ is the mixed numeral $6\frac{1}{6}$.

b) $\frac{17}{3}$ is the mixed numeral $5\frac{2}{3}$.

We can also use <u>fractions</u> to <u>justify</u> that a given mixed numeral has a particular fraction.

For example:

$$4\frac{2}{3} = 4 + \frac{2}{3}$$
$$= \frac{4}{1} + \frac{2}{3}$$
$$= \frac{4 \times 3}{1 \times 3} + \frac{2}{3}$$
$$= \frac{12}{3} + \frac{2}{3}$$
$$= \frac{14}{3}$$

all these steps are needed for the justification

3. Write out and complete the following example.

$$3\frac{5}{7} = \square + \square$$
$$= \frac{\square}{\square} + \frac{5}{7}$$
$$= \frac{3 \times \square}{1 \times \square} + \frac{5}{7}$$
$$= \frac{\square}{\square} + \frac{5}{7}$$
$$= \frac{\square}{\square}$$

(note: we want to write the whole number as a fraction with the same denominator as the fraction in the mixed numeral).

4. Justify that the following mixed numerals have the given fractions:

   a) $2\frac{3}{4}$ has the fraction $\frac{11}{4}$

   b) $6\frac{5}{8}$ has the fraction $\frac{53}{8}$

- Check your answers with those given at the end of the topic.
- Read objective B2.1 and I2.2.
- Do CHECK EXERCISE 2A.

CHECK EXERCISE 2A

B2.1 i) Write the following fractions as mixed numerals.

a) $\frac{34}{15}$ d) $\frac{10}{7}$

b) $\frac{48}{17}$ e) $\frac{17}{5}$

c) $\frac{27}{10}$

ii) Write the following mixed numerals as fractions.

a) $3\frac{5}{8}$ d) $1\frac{11}{12}$

b) $7\frac{5}{6}$ e) $5\frac{15}{22}$

c) $3\frac{7}{15}$

I2.2 i) Use fractions to justify that the mixed numeral for

a) $2\frac{8}{3}$ is $9\frac{1}{3}$

b) $\frac{20}{9}$ is $2\frac{2}{9}$

ii) Use fractions to justify that the fraction for

a) $8\frac{5}{6}$ is $\frac{53}{6}$

b) $7\frac{3}{4}$ is $\frac{31}{4}$

- Check your answers with those given at the end of the topic. You were successful on CHECK EXERCISE B2.1 if you had at least 4 parts correct in each of 2.1(i) and 2.1(ii). You were successful on CHECK EXERCISE I2.2 if you had 3 of the 4 parts correct and indicated all steps.

- If you are not sure how to work with mixed numerals or if you were not successful; check reference book - Modern School Mathematics pp. 345, 346 or ask a student helper or ask your teacher.
- Otherwise, go on to section 3.

OBJECTIVE B3.1

To find the sum of two or more rational numbers named by fractions or mixed numerals.

Example

Find the sums and write each as a mixed numeral with fraction part as a basic fraction.

A. $$\begin{array}{r} 4\frac{1}{8} \\ +\ 2\frac{1}{4} \\ \hline \end{array}$$

B. $5\frac{1}{2} + 2\frac{2}{3} + \frac{3}{4}$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A. $$\begin{array}{rcr} 4\frac{1}{8} & = & 4 + \frac{1}{8} \\ 2\frac{1}{4} & = & 2 + \frac{2}{8} \\ \hline & & 6\frac{3}{8} \end{array}$$

B. $$\begin{aligned} 5\frac{1}{2} + 2\frac{2}{3} + \frac{3}{4} &= 5\frac{6}{12} + 2\frac{8}{12} + \frac{9}{12} \\ &= 7 + \frac{23}{12} \\ &= 7 + 1\frac{11}{12} \\ &= 8\frac{11}{12} \end{aligned}$$

OBJECTIVE I3.1

To find the sum of two or more rational numbers named by fractions or mixed numerals.

Example

Find the sums and write each as a mixed numeral with fraction part as a basic fraction.

A. $$\begin{array}{r} 1\frac{1}{4} \\ \frac{2}{5} \\ +\ 1\frac{2}{3} \\ \hline \end{array}$$

B. $\frac{4}{5} + 3 + 4\frac{1}{3} + \frac{7}{10}$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A. $1\frac{1}{4} = 1 + \frac{15}{60}$ $\frac{2}{5} = \frac{24}{60}$ $1\frac{2}{3} = 1 + \frac{40}{60}$ $= 2 + \frac{79}{60} = 1 + 2 + \frac{19}{60} = 3\frac{19}{60}$	B. $\frac{4}{5} + 3 + 4\frac{1}{3} + \frac{7}{10} = \frac{24}{30} + 3 + 4\frac{10}{30} + \frac{21}{30}$ $= 7 + \frac{55}{30}$ $= 7 + 1\frac{25}{30}$ $= 8\frac{25}{30}$ $= 8\frac{5}{6}$

OBJECTIVE B3.2

To find the difference of two rational numbers named by fractions or mixed numerals.

Example

Find the differences and write the fraction parts as basic fractions.

A. $3\frac{2}{3} - 1\frac{2}{5}$

B. $$\begin{array}{r} 4\frac{1}{4} \\ -\ 3\frac{5}{6} \\ \hline \end{array}$$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A. $3\frac{2}{3} - 1\frac{2}{5} = 3\frac{10}{15} - 1\frac{6}{15}$ $= 2\ \frac{4}{15}$	B. $$\begin{array}{rcrcr} 4\frac{1}{4} & = & 4 + \frac{3}{12} & = & 3 + \frac{15}{12} \\ -\ 3\frac{5}{6} & = & 3 + \frac{10}{12} & = & 3 + \frac{10}{12} \\ \hline & & & & \frac{5}{12} \end{array}$$

OBJECTIVE I3.2

To find the difference of two rational numbers named by fractions or mixed numerals.

Example

Find the differences and write the fraction parts as basic fractions:

A. $3\frac{5}{8} - 2\frac{4}{5}$

B. $$\begin{array}{r} 6\frac{5}{9} \\ -\ \frac{3}{4} \\ \hline \end{array}$$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A.
$$\begin{aligned} 3\frac{5}{8} - 2\frac{4}{5} &= 3\frac{25}{40} - 2\frac{32}{40} \\ &= 1 + \frac{25}{40} - \frac{32}{40} \\ &= \frac{65}{40} - \frac{32}{40} \\ &= \frac{33}{40} \end{aligned}$$

B.
$$\begin{array}{rcrcr} 6\frac{5}{9} & = & 6 + \frac{20}{36} & = & 5 + \frac{56}{36} \\ -\ \underline{\frac{3}{4}} & = & \underline{\frac{27}{36}} & = & \underline{\frac{27}{36}} \\ & & & & 5\frac{29}{36} \end{array}$$

Section 3. Addition and Subtraction involving rational numbers named by mixed numerals

Example 1 Read through this example and the comments

$$
\begin{aligned}
& 3\tfrac{1}{4} + 2\tfrac{2}{3} \\
= \ & 3 + \tfrac{1}{4} + 2 + \tfrac{2}{3} \\
= \ & 3 + 2 + \tfrac{1}{4} + \tfrac{2}{3} \\
= \ & 5 + \tfrac{3}{12} + \tfrac{8}{12} \\
= \ & 5 + \tfrac{11}{12} \\
= \ & 5\tfrac{11}{12}
\end{aligned}
$$

Mixed numerals represent the sum of a number named by a whole number numeral and a number named by a fraction.

Add the whole numbers and add the numbers named by the fractions. The L.C.D. is 12.

Write the result as a mixed numeral.

You can probably do an example like this without writing down steps 2, 3 or 5

$$
\begin{aligned}
& 3\tfrac{1}{4} + 2\tfrac{2}{3} \\
= \ & 5 + \tfrac{3}{12} + \tfrac{8}{12} \\
= \ & 5\tfrac{11}{12}
\end{aligned}
$$

Example 2 Read through this example and the comments

$$
\begin{aligned}
& 3\tfrac{3}{4} + 2\tfrac{2}{3} \\
= \ & 5 + \tfrac{9}{12} + \tfrac{8}{12} \\
= \ & 5 + \tfrac{17}{12} \\
= \ & 5 + 1 + \tfrac{5}{12} \\
= \ & 6\tfrac{5}{12}
\end{aligned}
$$

Add the whole numbers and add the numbers named by the fractions. The L.C.D. is 12.

$\frac{17}{12}$ is $1\frac{5}{12}$ as a mixed numeral, i.e. $1 + \frac{5}{12}$

Add the whole numbers.

1. Find the sum: $5\frac{3}{4} + \frac{5}{6}$

Answer: 1:

$$
\begin{aligned}
& 5\tfrac{3}{4} + \tfrac{5}{6} \\
= \ & 5 + \tfrac{9}{12} + \tfrac{10}{12} \\
= \ & 5 + \tfrac{19}{12} \\
= \ & 5 + 1 + \tfrac{7}{12} \\
= \ & 6\tfrac{7}{12}
\end{aligned}
$$

<u>Example 3</u> Read through this example and the comment

$$4\tfrac{2}{3} - 1\tfrac{1}{2}$$

$$= 4 + \frac{2}{3} - (1 + \frac{1}{2})$$

$$= 4 - 1 + \frac{2}{3} - \frac{1}{2}$$

$$= 3 + \frac{4}{6} - \frac{3}{6}$$

$$= 3 + \frac{1}{6}$$

$$= 3\tfrac{1}{6}$$

Subtract the whole numbers and subtract the numbers named by the fractions. The L.C.D. is 6.

You can probably do an example like this without writing down steps 2, 3 or 5.

<u>Example 4</u> Read through this example and the comments

Sometimes in subtraction it is necessary to "borrow". This happens when the number to be subtracted is greater than the number it is to be subtracted from.

$$4\tfrac{1}{3} - 1\tfrac{3}{4}$$

$$= 3 + \frac{4}{12} - \frac{9}{12}$$

$$= 2 + 1\tfrac{4}{12} - \frac{9}{12}$$

$$= 2 + \frac{16}{12} - \frac{9}{12}$$

$$= 2 + \frac{7}{12}$$

$$= 2\tfrac{7}{12}$$

Subtract the whole numbers. When attempting to subtract the numbers named by fractions we see that $\frac{9}{12}$ is greater than $\frac{4}{12}$. "Borrow 1" from the whole number $1\frac{4}{12} = \frac{16}{12}$. Now we can subtract.

2. Subtract: $5\frac{3}{8} - 4\frac{5}{6}$

Answer: 2: $5\frac{3}{8} - 4\frac{5}{6}$

$$= 1 + \frac{9}{24} - \frac{20}{24}$$

$$= 1\tfrac{9}{24} - \frac{20}{24}$$

$$= \frac{33}{24} - \frac{20}{24}$$

$$= \frac{13}{24}$$

(addition may also be done with the numbers arranged vertically).

Examples 5 and 6 Read through these examples.

$$\begin{array}{rcl} 3\frac{1}{4} & = & 3 + \frac{3}{12} \\ 2\frac{1}{3} & = & 2 + \frac{4}{12} \\ + 4\frac{1}{6} & = & 4 + \frac{2}{12} \\ \hline & & 9 + \frac{9}{12} \\ & = & 9\frac{3}{4} \end{array}$$

L.C.D. is 12

Basic fraction for $\frac{9}{12}$ is $\frac{3}{4}$

$$\begin{array}{rcl} 3\frac{3}{4} & = & 3 + \frac{18}{24} \\ \frac{5}{6} & = & \frac{20}{24} \\ + 1\frac{7}{8} & = & 1 + \frac{21}{24} \\ \hline & & 4 + \frac{59}{24} \\ & = & 4 + 2 + \frac{11}{24} \\ & = & 6\frac{11}{24} \end{array}$$

$\frac{59}{24} = 2\frac{11}{24}$

3. Add:

$$\begin{array}{r} \frac{9}{10} \\ 2\ \frac{1}{4} \\ + 1\ \frac{2}{5} \\ \hline \end{array}$$

Example 7 Read through this example

$$\begin{array}{rcccccl} 12\frac{1}{3} & = & 12 + \frac{8}{24} & = & 11 + 1\frac{8}{24} & = & 11 + \frac{32}{24} \\ - 3\frac{5}{8} & = & 3 + \frac{15}{24} & = & 3 + \frac{15}{24} & = & 3 + \frac{15}{24} \\ \hline & & & & & & 8 + \frac{17}{24} \\ & & & & & = & 8\frac{17}{24} \end{array}$$

$\frac{15}{24}$ is greater than $\frac{8}{24}$.
We need to "borrow 1" from the 12.

4. Subtract

$$\begin{array}{r} 4\frac{1}{6} \\ - 3\frac{4}{5} \\ \hline \end{array}$$

Now read Objectives I3.1 and I3.2 and their examples. These tell you what you are expected to be able to do for this section. When ready, turn to and do CHECK EXERCISES 3.1 and 3.2.

Answers: 3:

$$\begin{array}{rcl} \frac{9}{10} & = & \frac{18}{20} \\ 2\frac{1}{4} & & 2 + \frac{5}{20} \\ + 1\frac{2}{5} & & 1 + \frac{8}{20} \\ \hline & & 3 + \frac{31}{20} \\ & = & 3 + 1 + \frac{11}{20} \\ & = & 4\frac{11}{20} \end{array}$$

4:

$$\begin{array}{rcccccl} 4\frac{1}{6} & = & 4 + \frac{5}{30} & = & 3 + 1\frac{5}{30} & = & 3 + \frac{35}{30} \\ - 3\frac{4}{5} & = & 3 + \frac{24}{30} & = & 3 + \frac{24}{30} & = & 3 + \frac{24}{30} \\ \hline & & & & & & \frac{11}{30} \end{array}$$

CHECK EXERCISE 3

I3.1 Find the sums and write each as a mixed numeral with fractional part as a basic fraction:

a) $3\frac{9}{10} + 2\frac{3}{8} + 1\frac{3}{4}$

b) $4\frac{2}{3} + 3\frac{1}{5} + 5\ \frac{7}{12}$

c) $4\frac{1}{3} + 2\frac{3}{10} + 2\frac{1}{2}$

d) $$\begin{array}{r} 1\frac{3}{4} \\ 2\frac{5}{12} \\ +\ \frac{7}{2} \\ \hline \end{array}$$

e) $$\begin{array}{r} 9\frac{1}{5} \\ \frac{1}{3} \\ +\ 8 \\ \hline \end{array}$$

I3.2 Find the differences and write the fractional parts as basic fractions:

a) $8\frac{5}{6} - 4\frac{3}{10}$

b) $7\frac{1}{6} - 4\frac{3}{4}$

c) $17\frac{5}{8} - 5\frac{1}{4}$

d) $$\begin{array}{r} 91\frac{4}{7} \\ -\ 39\frac{7}{8} \\ \hline \end{array}$$

e) $$\begin{array}{r} 45\frac{5}{9} \\ -\ 37\frac{5}{6} \\ \hline \end{array}$$

- Check your answers with those given at the end of the topic. You are successful in each CHECK EXERCISE if you are using the correct method and have no more than one part incorrect in each question.
- If you are not sure how to add mixed numerals or if you were unsuccessful with CHECK EXERCISE 3.1, read section 3 and do Activity exercises 3.1.
- If you are not sure how to subtract mixed numerals or if you were unsuccessful with CHECK EXERCISE 3.2 read section 3 and do Activity exercises 3.2.
- Otherwise go on to Section 4.

Activity Exercise 3.1

To add $4\frac{1}{5} + 3\frac{2}{5}$ we first write each mixed numeral as the sum

$4 + \frac{1}{5} + 3 + \frac{2}{5}$ of a whole number and a number named by a fraction

$4 + 3 + \frac{1}{5} + \frac{2}{5}$ We add the whole numbers and then the numbers named by the fractions.

$7 + \frac{3}{5}$ We write the result

$7\frac{3}{5}$

1. Add $5\frac{3}{7} + 3\frac{2}{7}$

Carefully read through the following examples and comments:

a) $*2\frac{3}{8} + 5\frac{3}{4} + 3\frac{5}{6}$

$= 2 + \frac{3}{8} + 5 + \frac{3}{4} + 3 + \frac{5}{6}$

$= 2 + 5 + 3 + \frac{3}{8} + \frac{3}{4} + \frac{5}{6}$ in order to add the numbers named by fractions we need to express them with their L.C.D. of 24.

$*= 10 + \frac{9}{24} + \frac{18}{24} + \frac{20}{24}$

$*= 10 + \frac{47}{24}$

$= 10 + 1\frac{23}{24}$ the mixed numeral for $\frac{47}{24}$ is $1\frac{23}{24}$

$= 10 + 1 + \frac{23}{24}$

$= 11 + \frac{23}{24}$

$*= 11\frac{23}{24}$ expressed as a mixed numeral

* When doing such an example you would probably only need to write down the lines marked with an *.

b) * $5\frac{5}{6} + 6\frac{1}{2}$

$= 11 + \frac{5}{6} + \frac{1}{2}$

$*= 11 + \frac{5}{6} + \frac{3}{6}$ L.C.D. is 6

$*= 11 + \frac{8}{6}$ $\frac{8}{6}$ is the mixed numeral $1\frac{2}{3}$

$= 11 + 1 + \frac{2}{6}$

$*= 12\frac{2}{6}$

$*= 12\frac{1}{3}$ $\frac{2}{6}$ is the basic fraction $\frac{1}{3}$

Note: Always express the mixed numeral with a BASIC FRACTION.

2. Add the following:

a) $3\frac{9}{10} + 4\frac{4}{5}$

b) $11\frac{7}{10} + 16\frac{3}{4} + 5\frac{1}{2}$

c) $7\frac{1}{4} + 3\frac{7}{8} + 4\frac{5}{6}$

d)
$$\begin{array}{r} 2\frac{1}{2} \\ \frac{3}{5} \\ + 1\frac{1}{2} \\ \hline \end{array}$$

e)
$$\begin{array}{r} 2\frac{3}{5} \\ 3\frac{4}{7} \\ + 5\frac{3}{10} \\ \hline \end{array}$$

Activity Exercise 3.2

$4\frac{5}{7} - 2\frac{3}{7}$

$= 4 + \frac{5}{7} - (2 + \frac{3}{7})$ subtract whole numbers and numbers named by fractions.

$= 4 - 2 + \frac{5}{7} - \frac{3}{7}$

$= 2 + \frac{2}{7}$

$= 2\frac{2}{7}$ express as a mixed numeral

1. Find the difference:

$8\frac{4}{5} - 5\frac{1}{5}$

In order to subtract $7\frac{3}{4} - 5\frac{1}{8}$ we need to express the fractions with their L.C.D.

Thus $7\frac{3}{4} - 5\frac{1}{8}$ (You can do several steps at once.)

$= 7 - 5 + \frac{6}{8} - \frac{1}{8}$ fractions expressed with L.C.D. of 8.

$= 2 + \frac{5}{8}$

$= 2\frac{5}{8}$ expressed as a mixed numeral.

2. Find the difference:

$5\frac{2}{3} - 2\frac{2}{5}$

Here is another example:

$$\begin{array}{r} 27\frac{1}{3} \\ -18\frac{5}{6} \\ \hline \end{array}$$

$$\begin{array}{r} 27\frac{1}{3} \\ -\ 18\frac{5}{6} \\ \hline \end{array} \quad = \quad \begin{array}{r} 27 + \frac{2}{6} \\ -\ 18 + \frac{5}{6} \\ \hline \end{array}$$

We have expressed the fractions in the mixed numerals with common denominators. But we find we cannot subtract $\frac{5}{6}$ from $\frac{2}{6}$. It is too big. Thus we must "borrow" one from the whole number.

$$27 + \frac{2}{6} = 26 + 1 + \frac{2}{6}$$

$$= 26 + 1\frac{2}{6}$$

$$= 26 + \frac{8}{6}$$

Now we can subtract $\frac{5}{6}$ from $\frac{8}{6}$. We have

Summary

Section 1

Addition and subtraction of rational numbers named by fractions is usually done by obtaining equivalent fractions with least common denominators and then adding or subtracting the numerators. Answers are given as basic fractions.

Section 2

A mixed numeral contains a whole number numeral and a fraction.

Each mixed numeral has a corresponding fraction

$$4\frac{3}{8} = (4 \times 8) + 3 \text{ eighths}$$
$$= 35 \text{ eighths}$$
$$= \frac{35}{8}$$

AND

Each fraction with numerator greater than denominator has a corresponding mixed numeral.

$$\frac{35}{8} = 8\overline{)35} = 4\frac{3}{8}$$

Section 3

In addition (or subtraction) of rational numbers named by mixed numerals, the whole numbers are added (or subtracted) and the numbers named by the fractions are added (or subtracted).

Section 4

A solution for a condition for equality is a number, which when put in place of the variable, makes the condition a true statement.

Conditions for equality involving addition (or subtraction) of rational numbers are solved by finding (related conditions with the variable alone on one side.

Section 5

Addition and subtraction of rational numbers are used to answer questions about every day situations. Most examples are done by

(1) finding the relationship between what is asked and what is given.

(2) obtaining the relationship in mathematical form and doing the indicated mathematical work to get a mathematical answer.

(3) Using the mathematical answer to answer the original question.

Section 6

$$\frac{a}{b} \text{ of } \frac{c}{d} = \frac{a \times c}{b \times d}$$

Section 7

Multiplication of rational numbers is done as follows:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

When mixed numerals or whole numbers are involved, their corresponding fractions are used. Answers are given as basic fractions. Reduction to basic fractions is done by dividing numerator and denominator by their common factors (cancelling).

Section 8

The reciprocal of a number is another number whose product with the given number is 1.

The reciprocal of a non-zero rational number is obtained by interchanging the numerator and denominator of the fraction for the rational number.

Zero has no reciprocal.

Section 9

To divide by a non-zero rational number, you multiply by its reciprocal.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

It is not possible to divide by 0.

Section 10

Conditions for equality involving multiplication of rational numbers can often be solved by finding related conditions involving division of rational numbers with the variable alone on one side.

Also, in conditions for equality in which the variable is multiplied by a number, the variable can be obtained alone on one side by multiplying both sides by the reciprocal of the number by which the variable is multiplied.

Section 11

To answer questions from everyday situations by mathematics we usually find the relationship between what is asked for and what is given. The mathematical form of this relationship is called the mathematical model for the situation. The mathematical model is often a condition for equality. The solution for this condition for equality is used to answer the original question.

Section 12

The rational numbers are closed for addition and multiplication. Addition and multiplication of rational numbers are commutative and associative.

0 and 1 are the identities for addition and multiplication respectively of rational numbers.

Multiplication of rational numbers is distributive over addition. Non-zero rational numbers have reciprocals. This is a property of rational numbers which whole numbers do not have.

Vocabulary

analyse - with respect to a question about an every day situation it means to find the relationship between what is asked for and what is given. (page) (section 5, p 2)

application - use of mathematics to answer a question about an everyday situation. (page) (section 5,p 1)

associative - the property of an operation which permits a change in the grouping of the numbers to not affect the result (page) (section 12,p 1)

binary - a binary operation is done with just two numbers at a time. (page) (section 12, p 4)

cancel - to divide numerator and denominator of a fraction by a common factor. (page) (section 7, p 3)

closed - the property of an operation which states that the number resulting from the operation is in the same set as the numbers the operation is done with. (page) (section 12, p 1)

commutative - the property of an operation which permits a change in the order of the numbers to not affect the result. (page) (section 12, p 1)

complex fraction - a fraction like $\dfrac{\frac{2}{3}}{\frac{3}{4}}$ which represents $\frac{2}{3} \div \frac{3}{4}$.

condition for equality - a mathematical statement involving a variable and the = symbol. (page) (section 4, p 1)

difference - the result of a subtraction. (page) (section 1, Objectives)

distributive - a property of multiplication and addition of numbers (page) (section 12, p 1)

L.C.D. - least common denominator. (page) (section 1, p 2)

mixed numeral - a numeral involving a whole number numeral and a fraction. (page) (section 2, p 1)

model - the mathematical form for a relationship. (page) (section 11, p 1)

operation - addition, subtraction, multiplication and division are examples of operations. (page) (Introduction)

product - the result of a multiplication. (page) (section 7, p 1)

property - a characteristic or quality. (page) (section 12, p 1)

quotient - the result of a division.

reciprocal - a number whose product with a given number is 1. (page) (section 8, p 1)

related condition - a condition with the same solution as the given condition. (page) (section 4, p 1)

replacement set - the set of numbers which a variable can represent; same as universe. (page) (section 4, p 1)

solution - a number which makes a condition a true statement. (page) (section 4, p 1)

solve - the process of finding the solution to a condition or problem. (page) (section 4, p 1)

sum - the result of an addition. (page) (section 4, p 1)

universe - the set of numbers which a variable can represent; same as replacement set. (page) (section 4, p 1)

variable - a letter which can represent any of the numbers in a given set. (page) (section 4, p 1)

Solutions - Topic 2 Phase I

CHECK EXERCISE 1

I1.1 a) $\frac{67}{50}$ b) $\frac{95}{48}$ c) $\frac{31}{15}$ d) $\frac{41}{20}$ e) $\frac{77}{30}$

I1.2 a) $\frac{1}{2}$ b) $\frac{3}{10}$ c) $\frac{5}{18}$ d) $\frac{27}{40}$ e) $\frac{11}{12}$

Activity exercise 1.1

1 a) $\frac{6}{7}$ b)i) $\frac{7}{9}$ ii) $\frac{7}{4}$ iii) $\frac{11}{10}$ iv) $\frac{4}{5}$ c)i) $\frac{5}{4}$ ii) $\frac{5}{6}$ iii) $\frac{44}{15}$
iv) $\frac{71}{63}$ v) $\frac{187}{70}$ d)i) $\frac{2}{3}$ ii) $\frac{9}{10}$ iii) $\frac{47}{40}$ e)i) 2 ii) $\frac{9}{8}$
iii) $\frac{53}{40}$ iv) $\frac{23}{15}$ v) $\frac{9}{4}$

Activity exercise 1.2

a)i) $\frac{1}{9}$ ii) $\frac{4}{5}$ iii) $\frac{2}{8}$ b)i) $\frac{25}{15}$ ii) $\frac{7}{24}$ iii) $\frac{11}{35}$
c)i) $\frac{3}{16}$ ii) $\frac{1}{4}$ iii) $\frac{1}{4}$ iv) $\frac{1}{12}$ v) $\frac{43}{35}$

CHECK EXERCISE 1A

I1.1 a) $\frac{8}{9}$ b) $\frac{71}{100}$ c) $\frac{7}{12}$ d) $\frac{5}{6}$ e) $\frac{23}{24}$

I1.2 a) $\frac{1}{3}$ b) $\frac{3}{10}$ c) $\frac{7}{24}$ d) $\frac{5}{48}$ e) $\frac{3}{10}$

CHECK EXERCISE 2

B2.1 i) a) $7\frac{2}{5}$ b) $2\frac{1}{4}$ c) $3\frac{2}{3}$ d) $1\frac{6}{7}$ e) $1\frac{7}{9}$

ii) a) $\frac{7}{4}$ b) $\frac{19}{7}$ c) $\frac{19}{11}$ d) $\frac{40}{9}$ e) $\frac{43}{8}$

I2.2 i) a)
$$\frac{15}{4} = \frac{12}{4} + \frac{3}{4} = \frac{12 \div 4}{4 \div 4} + \frac{3}{4} = \frac{3}{1} + \frac{3}{4} = 3 + \frac{3}{4} = 3\frac{3}{4}$$

b)
$$\frac{22}{7} = \frac{21}{7} + \frac{1}{7} = \frac{21 \div 7}{7 \div 7} + \frac{1}{7} = \frac{3}{1} + \frac{1}{7} = 3 + \frac{1}{7} = 3\frac{1}{7}$$

CHECK EXERCISE 4

I4.1 a) $p = \frac{1}{30}$ b) $a = \frac{5}{48}$ c) $n = \frac{1}{3}$ d) $b = 12\frac{11}{24}$ e) $a = 7\frac{27}{35}$

Activity Exercise 4.1

1.a) $n = 3\frac{1}{5}$ b) $b = 2\frac{1}{35}$ 2a) $a = 5\frac{11}{15}$ b) $b = 1\frac{7}{9}$

CHECK EXERCISE 4A

I4.1 a) $r = 1\frac{1}{8}$ b) $b = 4\frac{1}{3}$ c) $y = 6\frac{13}{20}$ d) $a = 15\frac{3}{10}$ e) $n = 5\frac{3}{16}$

CHECK EXERCISE 5

I5.1 1. 2 miles 2. $35\frac{7}{8}$ miles 3. $1\frac{11}{12}$ feet 4. $3\frac{1}{3}$ hours

Activity Exercise 5

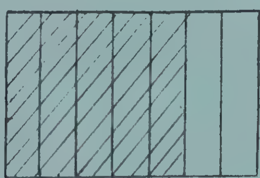
1. $\frac{103}{120}$ 2. $12\ \frac{1}{16}$ inches 3. $6\frac{1}{16}$ inches 4. $4\ \frac{85}{128}$ inches

CHECK EXERCISE 5A

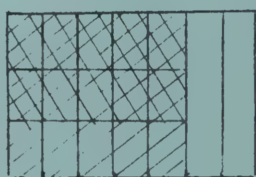
I5.1 1. $1\frac{1}{30}$ hours 2. $12\frac{1}{2}$ inches 3. $7\frac{15}{16}$ inches 4. up $2\frac{13}{24}$ points

CHECK EXERCISE 6

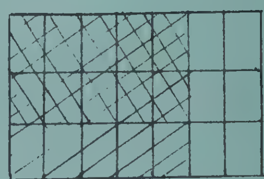
I6.1 i)



$\frac{5}{7}$ of region

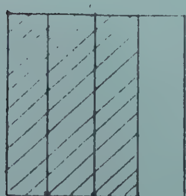


$\frac{2}{3}$ of $\frac{5}{7}$

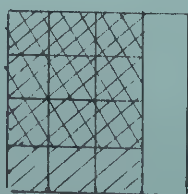


$\frac{2}{3}$ of $\frac{5}{7}$ is $\frac{10}{21}$

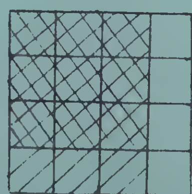
ii)



$\frac{3}{4}$ of region



$\frac{3}{4}$ of $\frac{3}{4}$



$\frac{3}{4}$ of $\frac{3}{4}$ is $\frac{9}{16}$

Review Exercises Topic II

Operations with rational numbers

You have completed the study of Topic II. It would be nice to know just how much you have learned in this topic. To do this you take a test, but before you take the test you should carefully review what you were expected to learn. To help you with this review we have prepared a set of exercises. If you have difficulty with any one of the exercises you should go back and review the appropriate objective (pink pages) and its development (white pages). The appropriate objective(s) for each exercise is (are) shown in the left margin next to the exercise.

I1.1 Find the sums and write each as a basic fraction:

A. $\frac{7}{9} + \frac{2}{5} + \frac{2}{3}$

B. $$\begin{array}{r} \frac{7}{10} \\ \frac{1}{5} \\ \frac{5}{12} \\ + \frac{1}{2} \\ \hline \end{array}$$

I1.2 Find the differences for the following and write each as a basic fraction:

A. $\frac{9}{10} - \frac{3}{4}$

B. $$\begin{array}{r} \frac{5}{12} \\ - \frac{3}{10} \\ \hline \end{array}$$

I2.1 A. Write a mixed numeral for each of the following fractions

i) $\frac{83}{6}$ ii) $\frac{257}{8}$

B. Write the following mixed numerals as fractions:

i) $6\frac{3}{5}$ ii) $15\frac{3}{8}$

I2.2 A. Justify that the mixed numeral for $\frac{43}{8}$ is $5\frac{3}{8}$.

B. Justify that the fraction for $7\frac{4}{5}$ is $\frac{39}{5}$.

I3.1 Find the following sums and write the fractional part as a basic fraction:

A. $3\frac{7}{8} + 2\frac{3}{4} + 1\frac{1}{2} + 6\frac{1}{4}$

B. $\begin{array}{r} 32\frac{3}{8} \\ 16\frac{1}{3} \\ +\ 15\frac{5}{6} \\ \hline \end{array}$

I3.2 Find the following differences and write the fractional part as a basic fraction:

A. $\begin{array}{r} 8\frac{4}{7} \\ -\ 3\frac{5}{8} \\ \hline \end{array}$

B. $4\frac{5}{12} - \frac{7}{9}$

I4.1 Solve each condition and write the fractions in the solutions as basic fractions:

A. $3\frac{5}{7} = n + 1\frac{3}{5}$

B. $n - \frac{3}{10} = 1\frac{5}{8}$

I5.1 John went on a hike. He covered a total distance of $2\frac{3}{16}$ miles. Yet when he came home he declared that he had walked only $1\frac{2}{3}$ miles. When questioned about this statement he said that he had run $\frac{25}{48}$ mile. Was John correct when he said that he had walked $1\frac{2}{3}$ miles?

I6.1 Show by diagrams how you can obtain $\frac{3}{4}$ of $\frac{4}{5}$.

I7.1 Find the basic fraction for each product:

A. $\frac{49}{66} \times \frac{27}{35} \times \frac{55}{63}$

B. $3 \times \frac{14}{75} \times 6\frac{3}{7}$

I8.1 i) Give an example that illustrates the property which a number and its reciprocal have.

ii) What is the reciprocal of 1?

iii) Write a whole number other than one and give its reciprocal.

iv) Give the rational number that has no reciprocal and explain why it has no reciprocal.

I9.1 Find the quotients of the following pairs of rational numbers:

A. $7\frac{5}{7} \div 3\frac{3}{14}$

B. $\dfrac{2\frac{1}{4}}{3\frac{3}{5}}$

I9.2 Explain in terms of reciprocals why $\frac{7}{12} \div 0$ is not possible.

I10.1 Solve each of the following conditions:

A. $6\frac{1}{4}\,a = 5\frac{5}{8}$ B. $1\frac{3}{4} = 2\frac{3}{16}\,n$

I11.1 A. A dealer bought a motor bike for \$1200. He added $\frac{1}{6}$ of this cost to determine his selling price. What was his selling price?

B. Two sides of a triangular flower bed are $1\frac{2}{3}$ yards and $2\frac{3}{4}$ yards long. The perimeter of the flower bed is 8 yards. What is the length of the third side?

C. An inch is about $2\frac{1}{2}$ centimeters. How many inches are there in $26\frac{1}{4}$ centimeters?

I12.1 Give a numerical example which illustrates what is meant by each of the following statements. (one example for each)

(a) the set of rational numbers is closed under addition.
(b) addition of rational numbers is commutative.
(c) addition of rational numbers is associative.
(d) the set of rational numbers has an identity element for addition.

I12.2 Show what is meant by each of the following statements by giving a numerical example for each.

(a) the set of rational numbers is closed under multiplication.
(b) multiplication of rational numbers is commutative.
(c) multiplication of rational numbers is associative.
(d) the set of rational numbers has an identity element for multiplication.
(e) rational numbers have the property that multiplication is distributive over addition.
(f) every non-zero rational number has a reciprocal.

I12.3 (a) State a property of multiplication of rational numbers which the whole numbers do not have for multiplication.

(b) Give an example to illustrate this property of the rational numbers and one to show that the whole numbers do not have it.

Answers to Review Exercises Topic II

I1.1 A. $\frac{83}{45}$ B. $\frac{109}{60}$

I1.2 A. $\frac{3}{20}$ B. $\frac{7}{60}$

B2.1 A. i) $13\frac{5}{6}$; ii) $32\frac{1}{8}$ B. i) $\frac{33}{5}$; ii) $\frac{123}{8}$

I2.2 A.
$$\begin{aligned}\frac{43}{8} &= \frac{40}{8} + \frac{3}{8}\\ &= \frac{40 \div 8}{8 \div 8} + \frac{3}{8}\\ &= \frac{5}{1} + \frac{3}{8}\\ &= 5 + \frac{3}{8}\\ &= 5\frac{3}{8}\end{aligned}$$

B.
$$\begin{aligned}7\frac{4}{5} &= 7 + \frac{4}{5}\\ &= \frac{7}{1} + \frac{4}{5}\\ &= \frac{7 \times 5}{1 \times 5} + \frac{4}{5}\\ &= \frac{35}{5} + \frac{4}{5}\\ &= \frac{39}{5}\end{aligned}$$

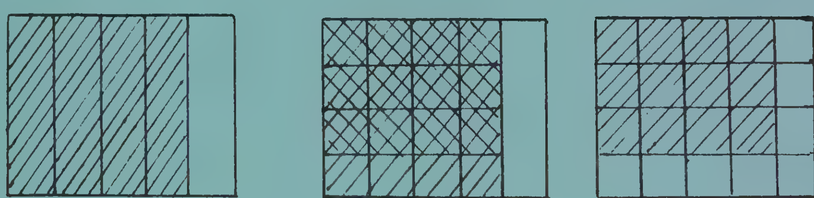
I3.1 A. $14\frac{3}{8}$ B. $64\frac{13}{24}$

I3.2 A. $4\frac{53}{56}$ B. $3\frac{23}{36}$

I4.1 A. $n = 2\frac{4}{35}$ B. $n = 1\frac{37}{40}$

I5.1 Yes

I6.1



$\frac{4}{5}$ $\frac{3}{4}$ of $\frac{4}{5}$ $\frac{3}{4}$ of $\frac{4}{5} = \frac{12}{20}$

I7.1 A. $\frac{1}{2}$ B. $\frac{18}{5}$ or $3\frac{3}{5}$

I8.1 i) $\frac{3}{4} \times \frac{4}{3} = 1$ any fraction can be used to illustrate this property

ii) 1

iii) can be any whole number, i.e. $17, \frac{1}{17}$.

iv) 0 has no reciprocal; $0 = \frac{0}{1}$ and there is no number by which $\frac{0}{1}$ can be multiplied to give 1, for $\frac{0}{1}$ times any number is 0.

OR. The reciprocal of $\frac{0}{1}$ is $\frac{1}{0}$, but $\frac{1}{0}$ has no meaning since we cannot divide by 0.

I9.1 A. $\frac{12}{5} = 2\frac{2}{5}$ B. $\frac{5}{8}$

I9.2 $\frac{7}{12} \div 0$ can be solved by converting to the corresponding multiplication and get $\frac{7}{12} \times \frac{1}{0}$, but $\frac{1}{0}$ does not exist and hence we cannot get the multiplication example. Therefore the division is not possible.

I10.1 A. $a = \frac{9}{10}$ B. $n = \frac{4}{5}$

I11.1 A. selling price is $1400 ; B. $3\frac{7}{12}$ yards; C. $10\frac{1}{2}$ inches

I12.1 Examples which show the same ideas as:

a) $\frac{3}{5} + \frac{5}{7} = \frac{46}{35}$ and $\frac{46}{35}$ is a rational number.

b) $\frac{3}{5} + \frac{5}{7} = \frac{5}{7} + \frac{3}{5}$

c) $(\frac{3}{5} + \frac{5}{7}) + \frac{7}{8} = \frac{3}{5} + (\frac{5}{7} + \frac{7}{8})$

d) $\frac{3}{5} + 0 = \frac{3}{5}$

I12.2 Examples which show the same ideas as:

a) $\frac{3}{5} \times \frac{5}{7} = \frac{3}{7}$ and $\frac{3}{7}$ is a rational number.

b) $\frac{3}{5} \times \frac{5}{7} = \frac{5}{7} \times \frac{3}{5}$

c) $(\frac{3}{5} \times \frac{5}{7}) \times \frac{7}{9} = \frac{3}{5} \times (\frac{5}{7} \times \frac{7}{9})$

d) $\frac{3}{5} \times 1 = \frac{3}{5}$

e) $\frac{3}{5} \times (\frac{5}{7} + \frac{7}{9}) = (\frac{3}{5} \times \frac{5}{7}) + (\frac{3}{5} \times \frac{7}{9})$

f) reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$.

I12.3 a) A statement which gives the same idea as: Every non-zero rational number has a reciprocal.

b) Examples which show the same idea as:

$\frac{3}{5}$ and $\frac{5}{3}$ are reciprocals for $\frac{3}{5} \times \frac{5}{3} = 1$.

3 has no reciprocal for there exists no whole number by which you can multiply 3 and get one as an answer.

TOPIC II (OPERATIONS WITH RATIONAL NUMBERS)

PHASE II

BASIC LEVEL

Students who achieved less than half of the objectives in Phase I are given this material. It is prepared to let you work at the level of difficulty which suits you. Using it will help you prepare for the next test.

On the next test, you will be expected to answer only those questions which relate to BASIC objectives that you did not achieve on the test you have just written. If you achieved an INTERMEDIATE objective, you will not have to work on the basic objective with the same number, (For example, if you got the question relating to objective I3.2 correct on the first test, then you have already achieved objective B3.2).

Use your record page to tell you which objectives you have not yet achieved. Use your new flow chart to guide you through Phase II and to keep a record of what you have done. Use the objectives in phase I materials to help you know what you have to learn. Use the following activities and exercises to practice your skills for the objectives you have not achieved.

BASIC LEVEL

OBJECTIVE B1

For each of the objectives in section 1 that you did not achieve do the following:

- Read the objective and the corresponding description.
- Do the appropriate exercises and CHECK EXERCISES on these pages.

B1.1 Addition of rational numbers named by fractions.

$\frac{5}{3} + \frac{1}{2} + \frac{4}{9}$

To do the addition at the left:

(1) Find the least (smallest) number which 3, 2 and 9 divide into; i.e. the least common multiple of 3, 2 and 9. It is 18. This is the least common denominator. (L.C.D.)

(2) For each fraction, find an equivalent fraction with the common denominator of 18.
$\frac{5}{3} = \frac{5 \times 6}{3 \times 6} = \frac{30}{18}$; $\frac{1}{2} = \frac{1 \times 9}{2 \times 9} = \frac{9}{18}$; $\frac{4}{9} = \frac{4 \times 2}{9 \times 2} = \frac{8}{18}$

$= \frac{30}{18} + \frac{9}{18} + \frac{8}{18}$

(3) Add the numerators.

$= \frac{47}{18}$

Find the following sums, and write each as a basic fraction:

1. $\frac{3}{4} + \frac{5}{6}$ (remember to find the least common denominator for each fraction and to form equivalent fractions with this L.C.D.)

2. $\frac{1}{3} + \frac{7}{9} + \frac{5}{6}$ (You must find the L.C.D. for all three fractions).

3. $\frac{3}{8} + \frac{1}{6} + \frac{5}{12}$

4. $\frac{2}{3} + \frac{1}{4} + \frac{5}{9}$

5. $\frac{3}{7} + \frac{1}{2} + \frac{5}{4}$

6. $$\begin{array}{rcl} \frac{3}{5} & = & \frac{\square}{\square} \\ \frac{1}{2} & = & \frac{\square}{\square} \\ +\frac{3}{4} & = & +\frac{\square}{\square} \\ \hline \end{array}$$

7. $$\begin{array}{r} \frac{7}{8} \\ \frac{3}{4} \\ +\ \frac{5}{16} \\ \hline \end{array}$$

B1.2 Subtraction of rational numbers named by fractions.

$\frac{5}{6} - \frac{3}{10}$

To do the subtraction at the left:

(1) Find the least (smallest) number which 6 and 10 divide into; i.e. the least common multiple of 6 and 10. It is 30. This is the least common denominator.

(2) For each fraction, find an equivalent fraction with the common denominator of 30.
$\frac{5}{6} = \frac{5 \times 5}{6 \times 5} = \frac{25}{30}$; $\frac{3}{10} = \frac{3 \times 3}{10 \times 3} = \frac{9}{30}$

$= \frac{25}{30} - \frac{9}{30}$

$= \frac{16}{30}$ (3) Subtract the numerators.

$= \frac{8}{15}$ (4) Give the answer as a basic fraction.

Find the differences, and write each as a basic fraction:

1. $\frac{3}{4} - \frac{1}{3}$ (remember to find the least common denominator for each fraction and to form equivalent fractions each with this denominator.)
2. $\frac{4}{5} - \frac{3}{4}$ (Again the L.C.D. must be found before subtracting).
3. $\frac{5}{8} - \frac{5}{12}$
4. $\frac{5}{6} - \frac{5}{8}$
5. $\frac{8}{9} - \frac{5}{6}$
6. $\frac{5}{8} = \boxed{-}$, $-\frac{1}{2} = \boxed{-}$
7. $\frac{11}{8} - \frac{3}{4}$

- Check your answers with those given at the end of the topic.
- Do the required parts in CHECK EXERCISE B1.

CHECK EXERCISE B1

B1.1 Find the sums and write each as a basic fraction:

a) $\frac{3}{4} + \frac{7}{8} + \frac{1}{2}$

b) $\frac{5}{2} + \frac{3}{4} + \frac{4}{5}$

c) $\frac{3}{8} + \frac{3}{4} + \frac{2}{3}$

d) $\frac{1}{2} + \frac{3}{8} + \frac{3}{4}$

e) $\frac{3}{4} + \frac{1}{12} + \frac{11}{6}$

B1.2 Find the differences and write each as a basic fraction:

a) $\frac{3}{4} - \frac{2}{3}$

b) $\frac{3}{2} - \frac{5}{8}$

c) $\frac{5}{9} - \frac{1}{2}$

d) $\frac{7}{6} - \frac{3}{4}$

e) $\frac{5}{2} - \frac{3}{5}$

- Check your answers with those given at the end of the topic.

  You were successful on CHECK EXERCISE

  B1.1 if you had at least 4 of the 5 parts correct.

  B1.2 if you had at least 4 of the 5 parts correct.
- If you are not certain how to add or subtract rational numbers, or if you were unsuccessful on a CHECK EXERCISE, consult your teacher. Then do the appropriate exercises that follow.
- Otherwise, go on to your next unachieved objective.

EXERCISES B1

B1.1 Find the sums and write each as a basic fraction:

a) $\frac{1}{2} + \frac{5}{6} + \frac{7}{12}$

b) $\frac{3}{8} + \frac{5}{12} + \frac{3}{4}$

c) $\frac{9}{16} + \frac{5}{8} + \frac{1}{2}$

d) $$\begin{array}{r} \frac{1}{4} \\ \frac{3}{5} \\ + \frac{3}{4} \\ \hline \end{array}$$

e) $$\begin{array}{r} \frac{5}{8} \\ \frac{5}{6} \\ + \frac{2}{3} \\ \hline \end{array}$$

B1.2 Find the differences and write each as a basic fraction:

a) $\frac{4}{5} - \frac{2}{5}$

b) $\frac{5}{8} - \frac{1}{2}$

c) $\frac{3}{5} - \frac{1}{4}$

d) $$\begin{array}{r} \frac{3}{4} \\ - \frac{2}{3} \\ \hline \end{array}$$

e) $$\begin{array}{r} \frac{5}{6} \\ - \frac{3}{4} \\ \hline \end{array}$$

- Check your answers with those given at the end of the topic.
- Go on to your next unachieved objective.

OBJECTIVE B2

- Read objective B2.1 and its description in section 2.
- Do the following exercises.

1. Write the following fractions as mixed numerals:

a) $\frac{10}{3}$ (remember: $\frac{10}{3} = \frac{9}{3} + \frac{1}{3} = 3 + \frac{1}{3} =$ _______)

or $\begin{array}{r} 3 \\ 3\overline{)\,10} \end{array}$ i.e. $3\frac{1}{3}$

b) $\frac{11}{3}$ (remember: $\frac{11}{3} = \frac{9}{3} + \frac{2}{3} =$ _____ = _____

or $\begin{array}{r} 3 \text{ rem. } 2 \\ 3\overline{)\,11} \end{array}$ i.e. $3\frac{2}{3}$)

c) $\frac{11}{8}$

d) $\frac{22}{7}$

e) $\frac{5}{2}$

f) $\frac{15}{4}$

g) $\frac{33}{10}$

2. Write the following mixed numerals as fractions:

a) $3\frac{1}{2}$ (remember: $3\frac{1}{2} = 3 + \frac{1}{2} = \frac{6}{2} + \frac{1}{2} =$ _______

or $3\frac{1}{2} = 3 + \frac{1}{2} =$ 6 halves + 1 half $= \frac{7}{2}$)

b) $2\frac{4}{5}$ (remember: $2\frac{4}{5} = 2 + \frac{4}{5} = \frac{10}{5} + \frac{}{} =$ _______

or $2\frac{4}{5} = 2 + \frac{4}{5} =$ 10 fifths + 4 fifths = 14 fifths $= \frac{14}{5}$

c) $1\frac{2}{3}$

d) $6\frac{2}{3}$

e) $7\frac{1}{2}$

f) $3\frac{3}{7}$

g) $1\frac{3}{8}$

- Check your answers with those given at the end of the topic.
- Do CHECK EXERCISE B2.

CHECK EXERCISE B2

B2.1 i) Write the following mixed numerals as fractions:

a) $3\frac{2}{5}$

b) $1\frac{5}{8}$

c) $2\frac{1}{2}$

d) $4\frac{4}{5}$

e) $3\frac{1}{4}$

B2.1 ii) Write the following fractions as mixed numerals:

a) $\frac{11}{4}$ d) $\frac{13}{3}$

b) $\frac{15}{8}$ e) $\frac{17}{8}$

c) $\frac{25}{4}$

- Check your answers with those given at the end of the topic.
- If you are unsure how to do the above exercises or if you had more than one part incorrect in either question (i) or question (ii); consult your teacher. Then do the following exercises.
- Otherwise, go on to your next unachieved objective.

EXERCISES B2

B2.1 i) Write the following mixed numerals as fractions:

a) $3\frac{1}{2}$ d) $4\frac{1}{4}$

b) $5\frac{3}{10}$ e) $1\frac{5}{6}$

c) $1\frac{3}{10}$

ii) Write the following fractions as mixed numerals:

a) $\frac{25}{8}$ d) $\frac{19}{7}$

b) $\frac{11}{6}$ e) $\frac{14}{5}$

c) $\frac{8}{3}$

- Check your answers with those given at the end of the topic.
- Go on to your next unachieved objective.

OBJECTIVE B3

For each of the objectives in section 3 that you did not achieve, do the following:

- Read the objective and corresponding description.
- Do the appropriate exercises and CHECK EXERCISES.

B3.1 Find the sums and write each as a mixed numeral with fractional part as a basic fraction:

a) $\begin{array}{r} 7\frac{1}{4} \\ +\ 2\frac{1}{2} \\ \hline \end{array}$ (Remember: $\begin{array}{rcl} 7\frac{1}{4} & = & 7 + \frac{1}{4} \\ +\ 2\frac{1}{2} & = & 2 + \frac{1}{2} \\ \hline & & 9 + \frac{3}{4} = \_\_\_\_ \end{array}$)

b) $7\frac{5}{6} + 4\frac{2}{3}$ (Remember: $7\frac{5}{6} + 4\frac{2}{3} = 7 + \frac{5}{6} + 4 + \frac{4}{6}$
$= 11 + \frac{9}{6}$
$= 11 + \_\_\_$
$= \_\_\_\_$)

c) $6\frac{1}{12} + 5\frac{1}{6} + 9\frac{2}{3}$

d) $8\frac{3}{5} + 13\frac{3}{4} + 3\frac{7}{10}$

e) $7\frac{1}{3} + 5\frac{3}{5} + 2\frac{1}{15}$

f) $\begin{array}{r} 4\frac{7}{12} \\ 9\ \frac{1}{2} \\ +\ 17\ \frac{3}{8} \\ \hline \end{array}$

g) $\begin{array}{r} 4\frac{5}{16} \\ 14\ \frac{1}{2} \\ +\ \ 3\ \frac{3}{8} \\ \hline \end{array}$

B3.2 Find the differences and write each as a mixed numeral with fractional part as a basic fraction.

a) $\begin{array}{rcl} 5\frac{3}{4} & = & 5 + \frac{9}{12} \\ -\ 2\frac{2}{3} & = & -2 + \_\_ \\ \hline & & 3 + \_\_\_ = \_\_\_ \end{array}$

b) $5\frac{1}{4} - 2\frac{3}{4} = \underset{(5-2)}{3} + \frac{1}{4} - \frac{3}{4}$
$= 2 + 1\frac{1}{4} - \frac{3}{4}$
$= 2 + \_\_ - \frac{3}{4}$
$= 2 + \_\_$
$= 2\frac{1}{2}$

c) $\begin{array}{r} 6\frac{3}{8} \\ -4\frac{3}{4} \\ \hline \end{array}$ d) $\begin{array}{r} 14\frac{4}{5} \\ -\ 13\frac{1}{2} \\ \hline \end{array}$ e) $19\frac{1}{2} - 12\frac{5}{7}$ f) $9 - 5\frac{2}{3}$

- Check your answers with those given at the end of the topic.
- Do the required parts in CHECK EXERCISE B3.

CHECK EXERCISE B3

B3.1 Find the sums and write each as a mixed numeral with fractional part as a basic fraction:

a) $6\frac{2}{3} + 4\frac{3}{4} + 5\frac{5}{6}$

b) $10\frac{3}{4} + 4\frac{1}{12} + 2\frac{5}{6}$

c) $7\frac{4}{5} + 8\frac{3}{4} + 2\frac{3}{10}$

d) $\begin{array}{r} 17\frac{1}{3} \\ 5\frac{5}{6} \\ +\ 4\frac{1}{2} \\ \hline \end{array}$ e) $\begin{array}{r} 3\ \frac{1}{8} \\ 2\frac{1}{16} \\ +\ 1\ \frac{1}{2} \\ \hline \end{array}$

B3.2 Find the differences and write each as a mixed numeral with fractional part as a basic fraction:

a) $5\frac{1}{2} - 3\frac{3}{8}$

b) $4\frac{3}{4} - 2\frac{1}{6}$

c) $17\frac{3}{4} - 6\frac{7}{8}$

d) $\begin{array}{r} 9\frac{1}{2} \\ -7\frac{5}{8} \\ \hline \end{array}$ e) $\begin{array}{r} 3\ \frac{3}{5} \\ -1\frac{9}{10} \\ \hline \end{array}$

- Check your answers with those given at the end of the topic. You were successful on CHECK EXERCISE
  B3.1 if you had at least 4 of the parts correct,
  B3.2 if you had at least 4 of the parts correct.
- If you are unsure how to add or subtract with mixed numerals, or if you were unsuccessful on either CHECK EXERCISE; consult your teacher. Then do the appropriate exercise that follows.
- Otherwise, go on to your next unachieved objective.

EXERCISES B3

B3.1 Find the sums and write each as a mixed numeral with fractional part as a basic fraction.

a) $3\frac{1}{12} + 5\frac{1}{6} + 7\frac{2}{3}$

b) $8\frac{1}{2} + 7\frac{3}{4} + 12\frac{5}{8}$

c) $7\frac{2}{3} + 11\frac{3}{4} + 8\frac{5}{9}$

d) $$\begin{array}{r} 11\frac{5}{9} \\ 8\frac{1}{3} \\ + \; 7\frac{5}{6} \\ \hline \end{array}$$

e) $$\begin{array}{r} 22\,\frac{2}{9} \\ 6\frac{5}{12} \\ + \; 4\,\frac{1}{4} \\ \hline \end{array}$$

B3.2 Find the differences and write each as a mixed numeral with fractional part as a basic fraction.

a) $6\frac{2}{3} - 2\frac{3}{8}$

b) $8\frac{1}{4} - 3\frac{2}{3}$

c) $15\frac{3}{8} - 9\frac{5}{12}$

d) $$\begin{array}{r} 7\frac{1}{2} \\ - \; 3\frac{2}{5} \\ \hline \end{array}$$

e) $$\begin{array}{r} 5\frac{1}{4} \\ - \; 2\frac{5}{6} \\ \hline \end{array}$$

- Check your answers with those given at the end of the topic.
- Go on to your next unachieved objective.

OBJECTIVE B4

- Read objective B4.1 and the corresponding description.
- Do the following exercises.

B4.1 Solution of conditions for equality in which the universe or replacement set for the variable is the set of rational numbers.

To solve the condition at the left:

$3\frac{4}{5} + n = 5\frac{1}{3}$

(1) form the corresponding condition involving subtraction.

$n = 5\frac{1}{3} - 3\frac{4}{5}$

(2) do the subtraction.

$= 2 + \frac{5}{15} - \frac{12}{15}$

$= 1 + \frac{20}{15} - \frac{12}{15}$

$= 1\frac{8}{15}$

(3) check the solution by replacing n.

Check: $3\frac{4}{5} + 1\frac{8}{15}$

$= 4 + \frac{12}{15} + \frac{8}{15}$

$= 4 + \frac{20}{15}$

$= 5\frac{5}{15}$

$= 5\frac{1}{3}$

(4) Give the solution.

Solution: $1\frac{8}{15}$

Solve the following conditions of equality and write fractions in solutions as basic fractions. The universe or replacement set for each variable is the set of rational numbers.

a) $7\frac{4}{5} + n = 8\frac{3}{4}$

form the corresponding condition involving subtraction.

$n = 8\frac{3}{4} - 7\frac{4}{5}$

$= 1 + \frac{15}{20} + \boxed{\frac{}{}}$

$= \frac{\boxed{}}{20} - \frac{16}{20}$

$n = \boxed{\frac{}{}}$

Check:

Solution:

Answers

OBJECTIVE B1

B1.1 1. $\frac{19}{12}$ 2. $\frac{35}{18}$ 3. $\frac{23}{24}$ 4. $\frac{53}{36}$ 5. $\frac{61}{28}$ 6. $\frac{12}{20}, \frac{10}{20}, \frac{15}{20}; \frac{37}{20}$
7. $\frac{31}{16}$

B1.2 1. $\frac{5}{12}$ 2. $\frac{1}{20}$ 3. $\frac{5}{24}$ 4. $\frac{5}{24}$ 5. $\frac{1}{18}$ 6. $\frac{5}{8}, \frac{4}{8}; \frac{1}{8}$ 7. $\frac{5}{8}$

CHECK EXERCISE B1

B1.1 a) $\frac{17}{8}$ b) $\frac{81}{20}$ c) $\frac{43}{24}$ d) $\frac{13}{8}$ e) $\frac{8}{3}$

B1.2 a) $\frac{1}{12}$ b) $\frac{7}{8}$ c) $\frac{1}{18}$ d) $\frac{5}{12}$ e) $\frac{19}{10}$

EXERCISES B.1

B1.1 a) $\frac{23}{12}$ b) $\frac{37}{24}$ c) $\frac{27}{16}$ d) $\frac{8}{5}$ e) $\frac{51}{24}$

B1.2 a) $\frac{2}{5}$ b) $\frac{1}{8}$ c) $\frac{7}{20}$ d) $\frac{1}{12}$ e) $\frac{1}{12}$

OBJECTIVE B2

B2.1 1. a) $3\frac{1}{3}$ b) $3\frac{2}{3}$ c) $1\frac{3}{8}$ d) $3\frac{1}{7}$ e) $2\frac{1}{2}$ f) $3\frac{3}{4}$ g) $3\frac{3}{10}$
2. a) $\frac{7}{2}$ b) $\frac{14}{5}$ c) $\frac{5}{3}$ d) $\frac{20}{3}$ e) $\frac{15}{2}$ f) $\frac{24}{7}$ g) $\frac{11}{8}$

CHECK EXERCISE B2

B2.1 i) a) $\frac{17}{5}$ b) $\frac{13}{8}$ c) $\frac{5}{2}$ d) $\frac{24}{5}$ e) $\frac{13}{4}$
ii) a) $2\frac{3}{4}$ b) $1\frac{7}{8}$ c) $6\frac{1}{4}$ d) $4\frac{1}{3}$ e) $2\frac{1}{8}$

EXERCISES B2

B2.1 i) a) $\frac{7}{2}$ b) $\frac{53}{10}$ c) $\frac{13}{10}$ d) $\frac{17}{4}$ e) $\frac{11}{6}$
ii) a) $3\frac{1}{8}$ b) $1\frac{5}{6}$ c) $2\frac{2}{3}$ d) $2\frac{5}{7}$ e) $2\frac{4}{5}$

OBJECTIVE B3

B3.1 a) $9\frac{3}{4}$ b) $12\frac{1}{2}$ c) $20\frac{11}{12}$ d) $26\frac{1}{20}$ e) 15 f) $31\frac{11}{24}$ g) $22\frac{3}{16}$

B3.2 a) $\frac{8}{12}, \frac{1}{12}, 3\frac{1}{12}$ b) $\frac{5}{4}, \frac{2}{4}$ c) $1\frac{5}{8}$ d) $1\frac{3}{10}$ e) $6\frac{11}{14}$ f) $3\frac{1}{3}$

CHECK EXERCISE B3

B3.1 a) $17\frac{1}{4}$ b) $17\frac{2}{3}$ c) $18\frac{17}{20}$ d) $27\frac{2}{3}$ e) $6\frac{11}{16}$

B3.2 a) $2\frac{1}{8}$ b) $2\frac{7}{12}$ c) $10\frac{7}{8}$ d) $1\frac{7}{8}$ e) $1\frac{7}{10}$

2B2B

EXERCISES B3

B3.1 a) $15\frac{11}{12}$ b) $28\frac{7}{8}$ c) $27\frac{35}{36}$ d) $27\frac{13}{18}$ e) $32\frac{8}{9}$

B3.2 a) $4\frac{7}{24}$ b) $4\frac{7}{12}$ c) $5\frac{23}{24}$ d) $4\frac{1}{10}$ e) $2\frac{5}{12}$

OBJECTIVE B4

B4.1 a) $\frac{16}{20}$, 35; $n = \frac{19}{20}$ b) $2\frac{2}{3} + 6\frac{1}{6}$, $\frac{4}{6} + \frac{1}{6}$; $n = 8\frac{5}{6}$ c) $n = 2\frac{3}{8}$
d) $n = 24\frac{5}{8}$ e) $n = 3\frac{11}{30}$ f) $n = 14\frac{1}{8}$

CHECK EXERCISE B4

B4.1 a) $n = 16\frac{11}{20}$ b) $n = 1\frac{1}{2}$ c) $n = 7\frac{1}{12}$ d) $n = 2\frac{2}{15}$ e) $n = 5\frac{1}{2}$

EXERCISES B4

B4.1 a) $n = 7\frac{6}{7}$ b) $n = 29\frac{1}{4}$ c) $n = 2\frac{13}{20}$ d) $n = 1\frac{3}{8}$ e) $n = 14\frac{1}{5}$

OBJECTIVE B5

B5.1 a) $n = 3\frac{5}{16} + 5\frac{13}{16} + \frac{1}{16}$, $\frac{5}{16} + \frac{13}{16} + \frac{1}{16}$, $\frac{19}{16}$, $9\frac{3}{16}$; $9\frac{3}{16}$ inches.
b) $497\frac{29}{72}$ lbs. c) $4\frac{7}{10}$ gal. d) $6\frac{5}{8}$ hr. e) $\frac{3}{8}$ inch f) $2\frac{7}{8}$ oz.

CHECK EXERCISE B5

B5.1 a) $14\frac{3}{8}$ lb. b) $6\frac{1}{8}$ yd. c) $10\frac{3}{8}$ in. d) $9\frac{1}{4}$ in. e) $\frac{1}{2}$ mi.

EXERCISES B5

B5.1 a) $\frac{5}{8}$ mi. b) $3\frac{1}{8}$ lb. c) $\frac{3}{8}$ lb. d) $1\frac{11}{12}$ lb. e) $3\frac{1}{10}$

OBJECTIVE B7

B7.1 a) $\frac{1}{5}$ b) $\frac{2 \times 9}{3 \times 1}$; 6 c) $\frac{13 \times 3}{9 \times 13}$; $\frac{1}{3}$ d) $\frac{3 \times 16}{4 \times 9}$; $1\frac{1}{3}$
e) 6 f) 33 g) $19\frac{4}{5}$ h) $\frac{1}{8}$ i) $9\frac{1}{6}$ j) 21

CHECK EXERCISE B7

B7.1 a) $\frac{7}{12}$ b) $7\frac{1}{2}$ c) 3 d) 45 e) 36

TOPIC II (OPERATIONS WITH RATIONAL NUMBERS)

PHASE II

INTERMEDIATE LEVEL

Students who achieved between half and nine tenths (90%) of the objectives in Phase I are given these materials.

Your job now is to master all of the objectives that you have missed.

On the next test, you will be expected to answer only those questions which are related to objectives that you did not achieve on the first test.

Use your record page to tell you which objectives you need to work on.

Use your flow chart to guide you through Phase II and to show what you have done in Phase I. You may want to do some of the parts you left out during Phase I.

Use your phase I materials to relearn the ideas for objectives you have not achieved and to give you examples of the type of questions you need to be able to answer.

Use these exercises to practice your skills so that you will be able to achieve all of the objectives.

INTERMEDIATE LEVEL

For each of the objectives you did not achieve on Post Test I, do the appropriate exercises and check your answers with those given at the end of the topic. Review the appropriate material in Phase I before starting each exercise.

OBJECTIVE I1.1

Find the following sums and write each as a basic fraction:

a) $\frac{5}{6} + \frac{3}{4} + \frac{4}{5}$

b) $\frac{3}{7} + \frac{1}{2} + \frac{2}{3}$

c) $\frac{5}{8} + \frac{5}{6} + \frac{5}{9}$

d) $\begin{array}{r} \frac{4}{9} \\ \frac{5}{12} \\ + \frac{2}{15} \\ \hline \end{array}$

e) $\begin{array}{r} \frac{4}{15} \\ \frac{3}{20} \\ + \frac{5}{6} \\ \hline \end{array}$

OBJECTIVE I1.2

Find the differences and write each as a basic fraction:

a) $\frac{8}{9} - \frac{3}{5}$

b) $\frac{4}{11} - \frac{1}{7}$

c) $\frac{9}{16} - \frac{3}{20}$

d) $\begin{array}{r} \frac{14}{25} \\ - \frac{4}{15} \\ \hline \end{array}$

e) $\begin{array}{r} \frac{17}{24} \\ - \frac{3}{16} \\ \hline \end{array}$

OBJECTIVE B2.1

i) Write the following mixed numerals as fractions:

a) $3\frac{1}{2}$

b) $5\frac{3}{10}$

c) $1\frac{3}{8}$

d) $4\frac{1}{4}$

e) $7\frac{5}{6}$

ii) Write the following fractions as mixed numerals:

a) $\frac{25}{8}$

b) $\frac{11}{6}$

c) $\frac{28}{3}$

d) $\frac{19}{7}$

e) $\frac{34}{5}$

OBJECTIVE I5.1

Solve the following applied problems:

a) Carol worked as a babysitter one summer. During one week she worked $3\frac{1}{2}$ hours on Monday, $4\frac{2}{3}$ hours on Tuesday, $2\frac{4}{5}$ hours on Wednesday, $5\frac{1}{6}$ hours on Saturday and $3\frac{3}{10}$ hours on Sunday. How many hours did Carol work during the week?

b) Four sections of a highway totaling $15\frac{2}{3}$ miles are to be built. Three of the sections measure $3\frac{1}{2}$, $5\frac{3}{5}$ and $5\frac{1}{6}$ miles. What is the length of the fourth section?

c) John, who weighed $135\frac{3}{4}$ pounds, went on a diet. After the first week he lost $2\frac{2}{3}$ pounds. However, during the second week he gained $1\frac{2}{5}$ pounds. How much did he weigh at the end of the second week?

d) A group of scouts joined short lengths of rope together in order to descend a steep bank. The lengths measured $3\frac{1}{2}$ ft., $2\frac{3}{3}$ 2 ft. 18 inches, and $4\frac{3}{8}$ ft. How many feet long was the joined rope? (Allow 1 foot for the knots.)

e) George and Bill leave their respective homes, $3\frac{2}{5}$ miles apart, planning to meet half-way. George walked $\frac{3}{4}$ of a mile then stopped to chat with a friend. He was still talking when Bill stopped for a bottle of pop $1\frac{1}{8}$ miles from his home. How far apart were the boys when they both stopped?

OBJECTIVE I6.1

a) Show by diagrams how $\frac{2}{3}$ of $\frac{3}{4}$ can be obtained.

b) Show by diagrams how $\frac{4}{5}$ of $\frac{2}{3}$ can be obtained.

OBJECTIVE I7.1

Find the following products:

a) $\frac{8}{9}$ of $7\frac{7}{8}$

b) $\frac{7}{8}$ of $\frac{4}{21}$

c) $\frac{14}{19} \times 3\frac{1}{7} \times 2\frac{3}{8}$

d) $\frac{4}{15} \times 22\frac{1}{2} \times 2\frac{2}{3}$

e) $2\frac{1}{4} \times 3\frac{1}{7} \times \frac{4}{27} \times 3$

OBJECTIVE I2.2

i) a) Use fractions to justify that the mixed numeral for $\frac{33}{7}$ is $4\frac{5}{7}$.

b) Use fractions to justify that the mixed numeral for $\frac{25}{3}$ is $8\frac{1}{3}$.

ii) a) Use fractions to justify that the fraction for $5\frac{2}{7}$ is $\frac{37}{7}$.

b) Use fractions to justify that the fraction for $6\frac{3}{7}$ is $\frac{45}{7}$.

OBJECTIVE I3.1

Find the following sums and write each as a mixed numeral with fractional part a basic fraction:

a) $9\frac{2}{3} + 13\frac{3}{4} + 7\frac{4}{5}$

b) $12\frac{7}{10} + 19\frac{3}{4} + 7\frac{7}{12}$

c) $4\frac{5}{16} + 17\frac{5}{9} + 10\frac{5}{12}$

d) $\begin{array}{r} 6\frac{7}{10} \\ 12\frac{4}{15} \\ +\ 2\frac{5}{6} \\ \hline \end{array}$

e) $\begin{array}{r} 11\frac{7}{9} \\ \frac{5}{6} \\ +\ 4\frac{3}{4} \\ \hline \end{array}$

OBJECTIVE I3.2

Find the differences and write the fraction parts of the mixed numerals as basic fractions:

a) $5\frac{7}{12} - 3\frac{8}{9}$

b) $11\frac{7}{9} - 3\frac{5}{6}$

c) $7 - 4\frac{2}{5}$

d) $\begin{array}{r} 8\frac{5}{24} \\ -\ 3\frac{7}{16} \\ \hline \end{array}$

e) $\begin{array}{r} 12\frac{11}{18} \\ -\ \frac{11}{12} \\ \hline \end{array}$

OBJECTIVE I4.1

Solve the following conditions of equality. The universe or replacement set for each variable is the set of rational numbers.

a) $3\frac{7}{9} = n + 2\frac{3}{4}$

b) $1\frac{1}{6} + p = 4\frac{1}{8}$

c) $m - 4\frac{5}{16} = 17\frac{5}{9}$

d) $12\frac{7}{15} = a - 9\frac{8}{9}$

e) $6\frac{3}{4} = 7\frac{2}{3} - z$

Answers

OBJECTIVE I1.1

a) $\frac{143}{60}$ b) $\frac{67}{42}$ c) $\frac{145}{72}$ d) $\frac{179}{180}$ e) $\frac{5}{4}$

OBJECTIVE I1.2

a) $\frac{13}{45}$ b) $\frac{17}{77}$ c) $\frac{33}{80}$ d) $\frac{22}{75}$ e) $\frac{25}{48}$

OBJECTIVE B2.1

i) a) $\frac{7}{2}$ b) $\frac{53}{10}$ c) $\frac{11}{8}$ d) $\frac{17}{4}$ e) $\frac{47}{6}$

ii) a) $3\frac{1}{8}$ b) $1\frac{5}{6}$ c) $9\frac{1}{3}$ d) $2\frac{5}{7}$ e) $6\frac{4}{5}$

OBJECTIVE I2.2

i) a) $\frac{33}{7} = \frac{28}{7} + \frac{5}{7} = \frac{4}{1} + \frac{5}{7} = 4 + \frac{5}{7} = 4\frac{5}{7}$

b) $\frac{25}{3} = \frac{24}{3} + \frac{1}{3} = \frac{8}{1} + \frac{1}{3} = 8 + \frac{1}{3} = 8\frac{1}{3}$

ii) a) $5\frac{2}{7} = 5 + \frac{2}{7} = \frac{35}{7} + \frac{2}{7} = \frac{37}{7}$

b) $6\frac{3}{7} = 6 + \frac{3}{7} = \frac{42}{7} + \frac{3}{7} = \frac{45}{7}$

OBJECTIVE I3.1

a) $31\frac{13}{60}$ b) $40\frac{1}{30}$ c) $32\frac{41}{144}$ d) $21\frac{4}{5}$ e) $17\frac{13}{36}$

OBJECTIVE I3.2

a) $1\frac{25}{36}$ b) $7\frac{17}{18}$ c) $2\frac{3}{5}$ d) $4\frac{37}{48}$ e) $11\frac{25}{36}$

OBJECTIVE I4.1

a) $n = 1\frac{1}{36}$ b) $p = 2\frac{23}{24}$ c) $m = 21\frac{125}{144}$ d) $a = 22\frac{16}{45}$ e) $z = \frac{11}{12}$

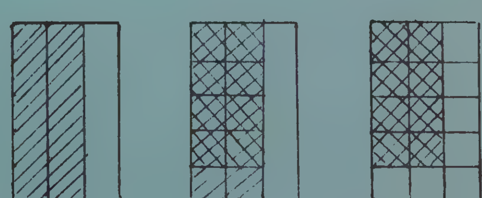
OBJECTIVE I5.1

a) $19\frac{13}{30}$ hr. b) $1\frac{2}{5}$ mi. c) $134\frac{29}{60}$ lb. d) $11\frac{1}{24}$ ft. e) $1\frac{21}{40}$ miles

OBJECTIVE I6.1

a)

b)



OBJECTIVE I7.1

a) 7 b) $\frac{1}{6}$ c) $5\frac{1}{2}$ d) 16 e) $3\frac{1}{7}$

OBJECTIVE I8.1

i) The product of a given number and its reciprocal is one.

ii) a) $\frac{3}{2}$ b) $\frac{1}{4}$ c) 1 d) 2 e) $\frac{2}{7}$

iii) Zero does not have a reciprocal as there is no number which multiplied by 0 gives a product of 1.

OBJECTIVE I9.1

a) $4\frac{2}{3}$ b) 45 c) $3\frac{4}{7}$ d) $\frac{27}{64}$ e) 0 f) 24 g) $\frac{3}{4}$

OBJECTIVE I9.2

$\frac{7}{9} \div \frac{0}{3}$ is not possible as to divide, we replace the division by multiplication by the reciprocal of the divisor, and $\frac{0}{3}$ does NOT have a reciprocal.

OBJECTIVE I10.1

a) $m = 22$ b) $p = 2\frac{1}{4}$ c) $q = 2\frac{2}{3}$ d) $n = 1\frac{11}{14}$ e) $a = 1\frac{35}{36}$

OBJECTIVE I11.1

a) 9 bags b) 260 mi. c) $4\frac{5}{24}$ d) 5 cups

e) $51\frac{9}{20}$ dollars f) $3\frac{3}{5}$

OBJECTIVE I12.1

a) $\frac{1}{2} + (\frac{2}{3} + \frac{1}{2}) = (\frac{1}{2} + \frac{2}{3}) + \frac{1}{2}$

b) $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$

c) $\frac{2}{5} + 0 = \frac{2}{5}$

d) $\frac{3}{4} + \frac{1}{3} = \frac{1}{3} + \frac{3}{4}$

or any similar answers

OBJECTIVE I12.2

i) a) commutative b) associative c) identity d) every non-zero rational number has a reciprocal e) closure f) zero product.

ii) distributive property of multiplication over addition.

OBJECTIVE I12.3

i) every non-zero rational number has a reciprocal.

ii) 5 x __ = 1 - there is no number to multiply by 5 to get a product of 1.

TOPIC II (OPERATIONS WITH RATIONAL NUMBERS)

PHASE II

ADVANCED LEVEL

Students who achieved nine tenths (90%) or more of the objectives for Phase I are given this material.

First, check your record page. If you missed any of the objectives there, go back through the materials and relearn the proper sections. You will be expected to answer the questions that relate to those objectives on the next test.

In this packet, there are several new objectives that you will be asked to achieve. Since there are only a few of these objectives, they have been collected at the beginning of the section. You should find them more interesting and challenging than the phase I objectives. You will be expected to answer questions on these objectives on the next test.

2A1

OBJECTIVE A1

To state the definition of a non-negative rational number.

Criterion: Statement to include the same ideas as:
"A non-negative rational number is a number named by a fraction of the form $\frac{a}{b}$ where a is a whole number and b is a non-zero whole number."

OBJECTIVE A2

To write the definitions for addition and subtraction of rational numbers named by fractions.

Criterion: Statements including the same ideas as:
"For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} ,"$$

OBJECTIVE A3

To write the definition for multiplication of rational numbers named by fractions.

Criterion: Statement including the same ideas as:
For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} ."$$

OBJECTIVE A4

To write the definition for division of rational numbers named by fractions.

Criterion: Statement including the same ideas as:
"For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ $(\frac{c}{d} \neq 0)$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

OBJECTIVE A5

To write the statements or reasons which complete the proof of a given property of operations for rational numbers.

Example:

For each of the numbered spaces in the proof below write the statement or reason which is missing.

Prove that addition of rational numbers is commutative:

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers.

Prove: $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$

Proof:

Statement	Reason
$\frac{a}{b} + \frac{c}{d}$	
$= \_\_\_\_\_\_$ (1)	Definition of addition of rational numbers
$= \frac{da + cb}{db}$	numbers $\_\_\_\_\_\_$ (2)
$= \frac{\_\_ + \_\_}{db}$ (3)	Addition of whole numbers is commutative
$= \frac{c}{d} + \frac{a}{b}$	$\_\_\_\_\_\_$ (4)
$\therefore \frac{a}{b} + \frac{c}{d} = \frac{c}{a} + \frac{a}{b}$	

i.e. Addition of rational numbers is commutative.

Criterion: 75% of computations correct.

SOLUTION

(1)	$\frac{ad + bc}{bd}$
(2)	Multiplication of rational numbers is commutative.
(3)	cb + da
(4)	Definition of addition of rational numbers.

2A3

OBJECTIVE A6

To use the distributive property to obtain products of rational numbers.

Example

a) Show how the distributive property may be used to simplify

$$(\frac{2}{3} \times \frac{2}{5}) + (\frac{7}{3} \times \frac{2}{5})$$

b) Show how the distributive property may be used to find the product

$$\frac{1}{2} \times 8\frac{2}{3}$$

c) Find the following product without changing the mixed numerals to fractions:

$$\begin{array}{r} 3\frac{1}{4} \\ \times\ 8\frac{2}{3} \\ \hline \end{array}$$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

a) $(\frac{2}{3} \times \frac{2}{5}) + (\frac{7}{3} \times \frac{2}{5}) = (\frac{2}{3} + \frac{7}{3}) \times \frac{2}{5} = \frac{9}{3} \times \frac{2}{5} = 3 \times \frac{2}{5} = \frac{6}{5}$

b) $\frac{1}{2} \times 8\frac{2}{3} = \frac{1}{2} \times (8 + \frac{2}{3}) = (\frac{1}{2} \times 8) + (\frac{1}{2} \times \frac{2}{3}) = 4 + \frac{1}{3} = 4\frac{1}{3}$

c)

$$\begin{array}{r} 3\frac{1}{6} \\ \times\ 4\frac{2}{3} \\ \hline 2\frac{2}{18} \\ 12\ \frac{4}{6} \\ \hline 14\frac{14}{18} \end{array} = 14\frac{7}{9}$$

1. DEFINITION FOR RATIONAL NUMBERS

In mathematics, a definition is a statement which tells precisely what something means.

In this section we will see the definition for rational numbers.

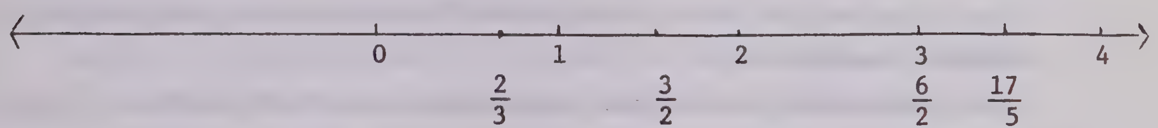
You have seen that each rational number is associated with an infinite set of equivalent fractions each of which has the form $\frac{a}{b}$ where a is a whole number and b is a non-zero whole number (or natural number).

e.g. The rational number two thirds is associated with the infinite set of fractions $\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\}$

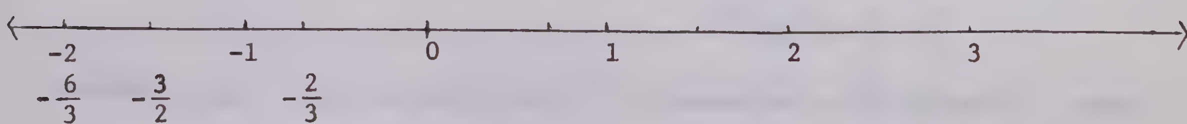
This gives us a way of defining rational numbers as follows:

<u>A rational number is a number named by a fraction of the form $\frac{a}{b}$ where a is a whole number and b is a non-zero whole number.</u>

This defines the sort of rational number which we have been using. That is, 0 and the rational numbers associated with points to the right of the point associated with 0 on the number line.



There are however rational numbers associated with points to the left of the point associated with 0 on the number line. Some of these are shown below.



Note the symbols used for these new rational numbers.

The two types of rational numbers are distinguished by calling those associated with points to the right of the point associated with 0 the POSITIVE RATIONAL NUMBERS and those associated with points to the left of the point associated with 0 the NEGATIVE RATIONAL NUMBERS.

You will learn more about the negative rational numbers later.

0 is neither a positive rational number nor a negative rational number.

0 and the positive rational numbers make up a set called the set of non-negative rational numbers (i.e. the rational numbers which are not negative).

The set of rational numbers we defined earlier is actually this set of non-negative rational numbers.

2. DEFINITIONS FOR ADDITION AND SUBTRACTION OF RATIONAL NUMBERS

In this section we will give definitions of addition and subtraction of rational numbers named by fractions; i.e. mathematical statements which give precisely the meaning of addition and subtraction of rational numbers named by fractions.

Apart from stating precisely what something means, a definition is a general statement which represents all possible particular instances of the thing being defined.

Let's see how the above ideas relate to addition and subtraction of rational numbers.

The definitions for addition and subtraction of rational numbers named by fractions are:

"For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}"$$

Note: ad means a x d, bc means b x c and bd means b x d. Pairs of letters written together as in the definitions indicate multiplication.

These may not look quite like the addition or subtraction we did in SECTION 1, however, they are general statements and do give the result for any addition or subtraction example.

1. Check that the sum of $\frac{5}{6}$ and $\frac{2}{5}$ is the same by the definition and by the common denominator method.
2. Check that the difference of $\frac{5}{6}$ and $\frac{3}{4}$ is the same by the definition and by the common denominator method.

Using the definition, you may not get the result as a basic fraction. However, a fraction and its equivalent basic fraction do name the same rational number.

Either the definition, or the common denominator method can be used to find the sum or difference of two rational numbers. We usually use the common denominator method (least common denominator in fact). However this method cannot be readily stated in the form of a general definition.

3. DEFINITION OF MULTIPLICATION OF RATIONAL NUMBERS

The definition for multiplication of rational numbers named by fractions is as given in OBJECTIVE A7.1.

As with the definitions for addition and subtraction of rational numbers named by fractions, the product may not be a basic fraction. However, we saw in SECTION 7 how the product can be reduced to a basic fraction.

Answers:

1. $\frac{5}{6} + \frac{2}{5} = \frac{5 \times 5 + 6 \times 2}{6 \times 5} = \frac{25 + 12}{30} = \frac{37}{30}$

   $\frac{5}{6} + \frac{2}{5} = \frac{25}{30} + \frac{12}{30} = \frac{37}{30}$

2. $\frac{5}{6} - \frac{3}{4} = \frac{5 \times 4 - 6 \times 3}{6 \times 4} = \frac{20 - 18}{24} = \frac{2}{24} = \frac{1}{12}$

   $\frac{5}{6} - \frac{3}{4} = \frac{10}{12} - \frac{9}{12} = \frac{1}{12}$

4. DEFINITION OF DIVISION OF RATIONAL NUMBERS

The definition for division of rational numbers named by fractions is as given in OBJECTIVE A9.1. Note the condition that the divisor cannot be zero since we cannot divide by 0.

5. PROVING PROPERTIES OF OPERATIONS WITH RATIONAL NUMBERS

In SECTION 12, we used examples to check that various properties of addition and multiplication of rational numbers named by fractions held.

We can in fact prove that these properties hold by using

(1) our knowledge of the properties of addition and multiplication of whole numbers (stated in SECTION 12)

and (2) the definitions for addition and multiplication of rational numbers named by fractions (stated in OBJECTIVES A1.1 and A7.1).

Let's see how two of the properties can be proved. Then you can complete the proofs of the others.

I Prove that addition of rational numbers is associative.

Let $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ be any rational numbers.

Prove: $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$

Proof:

Statement	Reason	Comment
$(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{(ad + bc)}{bd} + \frac{e}{f}$	definition of addition	
$= \frac{(ad + bc)f + (bd)e}{(bd)f}$	definition of addition	(bd)e means (b x d)xe
$= \frac{f(ad + bc) + (bd)e}{(bd)f}$	Mult. of whole numbers is commutative	Change of order.
$= \frac{\{f(ad) + f(bc)\} + (bd)e}{(bd)f}$	distributive property for whole numbers	
$= \frac{\{(ad)f + (bc)f\} + (bd)e}{(bd)f}$	Mult. of whole numbers is commutative	Change of order.
$= \frac{(ad)f + (bc)f + (bd)e}{(bd)f}$	Add. of whole numbers is associative	Grouping for addition does not matter.
$= \frac{adf + bcf + bde}{bdf}$	Mult. of whole numbers is associative.	Grouping for multiplication does not matter.

2A8

$\frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$		
$= \frac{a}{b} + \frac{(cf + de)}{df}$	definition of addition	
$= \frac{a(df) + b(cf + de)}{b(df)}$	definition of addition	
$= \frac{a(df) + \{b(cf) + b(de)\}}{b(df)}$	distributive property for whole numbers.	
$= \frac{a(df) + b(cf) + bde}{b(df)}$	Add. of whole numbers is associative.	
$= \frac{adf + bcf + bde}{bdf}$	Mult. of whole numbers is associative.	
$\therefore (\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$		The last line in each set of equalities is the same.

$\therefore$ Addition of rational numbers is associative.

$\therefore$ is an abbreviation for 'therefore'. However, it is never used in an English sentence.

A proof, as you see above, is a sequence of statements for each of which a reason is given. Proofs can often be conveniently given in two column form. The first column gives the statements; the second column gives the reasons for the statement.

In our proof above, the reasons used are the definition of addition for rational numbers and the properties of operations for whole numbers.

II Prove that 1 is the identity for multiplication of rational numbers.

Let $\frac{a}{b}$ be any rational number.

Prove: $\frac{a}{b} \times 1 = \frac{a}{b}$

Proof:

Statement	Reason
$\frac{a}{b} \times 1$	$\frac{1}{1}$ is another name for 1.
$= \frac{a}{b} \times \frac{1}{1}$	definition of multiplication
$= \frac{a \times 1}{b \times 1}$	
$= \frac{a}{b}$	1 is the identity for multiplication of whole numbers.

$\therefore$ 1 is the identity for multiplication of rational numbers.

Now turn to and do the activities on the following pages.

SECTION 5 - ACTIVITY

Each of the following is a proof for one of the properties of operations with rational numbers. In these proofs, certain statements or reasons have been omitted and left for you to give. Each place where something has been omitted is marked with an * and numbered. Write your answers in your workbook, NOT on these pages. At the end of each proof, check the answers you gave with the answers given at the end of the advanced work. If you cannot see why a particular answer is given, ask another person in the A group or ask your teacher.

The symbol "∴" means "therefore".

1. Prove that the rational numbers are closed for addition.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers.

Prove: $\frac{a}{b} + \frac{c}{d}$ is a rational number.

Proof:

Statement	Reason	Comment
$\frac{a}{b} + \frac{c}{d}$		
$= \frac{ad + bc}{bd}$	Definition of addition of rational numbers	
ad and bc are whole numbers	Whole numbers are closed for multiplication	Here we make use of properties of whole numbers.
(ad + bc) is a whole number	________ *1	
bd is a non-zero whole number	$b \neq 0$, $d \neq 0$ and ____ *2	
∴ $\frac{ad + bc}{bd}$ is a rational number	Definition of rational number.	
i.e. $\frac{a}{b} + \frac{c}{d}$ is a rational number		This comes from the first two lines in the proof.
i.e. The rational numbers are closed for addition.		

2A10

2. Prove that the rational numbers are closed for multiplication.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers.

Prove: $\frac{a}{b} \times \frac{c}{d}$ is a rational number.

Proof:

Statement	Reason	Comment
$\frac{a}{b} \times \frac{c}{d}$		
$= \frac{ac}{bd}$	______________ *3	ac means a x c
ac is a whole number	______________ *4	
______________ *5	$b \neq 0$, $d \neq 0$, and whole numbers are closed for multiplication.	The denominator must be non-zero.
$\therefore \frac{ac}{bd}$ is a rational number.	______________ *6	
i.e. $\frac{a}{b} \times \frac{c}{d}$ is a rational number.		This comes from the first two lines of the proof.
i.e. The rational numbers are closed for multiplication.		

3. Prove that multiplication of rational numbers is commutative.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers.

Prove: $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$

Proof:

Statement	Reason	
$\frac{a}{b} \times \frac{c}{d}$		
$= \frac{ac}{bd}$	Definition of multiplication of rational numbers	
$= \frac{ca}{db}$	______________ *7	The order is changed.
$=$ ______________ *8	Definition of multiplication of rational numbers.	← Here the definition of multiplication of rational numbers is used in the reverse direction to its use in the first two lines of the proof.
$\therefore \frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$		
i.e. ______________ *9		

4. Prove that addition of rational numbers is commutative.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers.

Prove: $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$

Proof:

Statement	Reason	Comment
$\frac{a}{b} + \frac{c}{d}$		
= ________________ *10	Definition of addition of rational numbers.	
$= \frac{da + cb}{db}$	________________ *11	This property is used three times here.
= ________________ *12	Addition of whole numbers is cummutative.	
$= \frac{c}{d} + \frac{a}{b}$	________________ *13	
∴ __________ = ______ *14		
i.e. Addition of rational numbers is commutative		

5. Prove that multiplication of rational numbers is associative.

Let $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ be any rational numbers.

Prove: $(\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f} = \frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f})$

Proof:

Statement	Reason	Comment
$(\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f}$		
$= (\frac{ac}{bd}) \times \frac{e}{f}$	________________ *15	The definition is used twice here.
$= \frac{(ac)\ e}{(bd)\ f}$	Definition of multiplication of rational numbers.	
$= \frac{(\quad)}{(\quad)}$ *16	Mult. of whole numbers is associative.	The grouping is changed.
$= \frac{a}{b} \times (\frac{cd}{df})$	________________ *17	
= ____________ *18	Definition of mult. of rational numbers	
$\therefore (\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f} = \frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f})$		
i.e. ______________________ *19		

Note: In proof I on page of the description we proved that addition of rational numbers is associative.

6. Prove that 0 is the identity for addition of rational numbers.

Let $\frac{a}{b}$ be any rational number.

Prove: $\frac{a}{b} + 0 = \frac{a}{b}$

Proof:

Statement	Reason	Comment
$\frac{a}{b} + 0$		
$= \frac{a}{b} + \frac{0}{1}$		$\frac{0}{1}$ is another name for 0.
$= \frac{a \times 1 + b \times 0}{b \times 1}$	________________ *20	
$= \frac{\quad + \quad}{}$ *21	1 is the identity for mult. of whole numbers and $b \times 0 = 0$	
$= \frac{a}{b}$	________________ *22	Another property of whole numbers.
$\therefore$ ____________ *23		
i.e. 0 is the identity for addition of rational numbers.		

7. Prove that multiplication is distributive over addition of rational numbers.

Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers.

Prove: $\frac{a}{b} \times (\frac{c}{d} + \frac{e}{f}) = (\frac{a}{b} \times \frac{c}{d}) + (\frac{a}{b} \times \frac{e}{f})$

Proof:

Statement	Reason	Comment
$\frac{a}{b} \times (\frac{c}{d} + \frac{e}{f})$		
$= \frac{a}{b} \times (\text{———})$ *24	Definition of addition of rational numbers.	
$= \frac{a\ (cf + de)}{b(df)}$	________________ *25	
$= \frac{a(cf) + a(de)}{b(df)}$	________________ *26	
$= \frac{acf + ade}{bdf}$	Multiplication of whole numbers is associative.	This permits the parentheses to be omitted

$(\frac{a}{b} \times \frac{c}{d}) + (\frac{a}{b} \times \frac{e}{f})$		Now we start with the right side of what we have to prove and get it into the same form as that to which we changed the left side.
$= \frac{ac}{bd} + \frac{ae}{bf}$	________________ *27	
$= \frac{(ac)(bf) + (bd)(ae)}{(bd)(bf)}$	________________ *28	
$= \frac{acbf + bdae}{bdbf}$	________________ *29	
$= \frac{bacf + bdae}{bdbf}$	Mult. of whole numbers is commutative.	The order in the products in the numerator has been changed.
$= \frac{b(acf + dae)}{bdbf}$	Distributive property for whole numbers.	
$= \frac{b(acf + dae)}{bdbf}$	Numerator and denominator are divided by b.	The fraction is reduced.
$= \frac{acf + dae}{dbf}$		
$= \frac{acf + ade}{bdf}$	________________ *30	The order is changed in two of the products This is now the same as the left side.
$\therefore \frac{a}{b} \times (\frac{c}{d} + \frac{e}{f}) =$ __________ *31		
i.e. Multiplication is distributive over addition of rational numbers.		

8. Prove that the multiplication property of zero holds for the rational numbers.

Let $\frac{a}{b}$ be any rational number.

Prove: $\frac{a}{b} \times 0 = 0$

Proof:

Statement	Reason	Comment
$\frac{a}{b} \times 0$		
$= \frac{a}{b} \times \frac{0}{1}$		$\frac{0}{1}$ is another name for 0
$=$ ______ *32	Definition of mult. of rational numbers.	
$= \frac{0}{b}$	__________ and ________ *33	$\frac{0}{b}$ is another name for 0.
$= 0$		
$\therefore \frac{a}{b} \times 0 = 0$		
i.e. ________________ *34		

6. USE OF THE DISTRIBUTIVE PROPERTY

Multiplication and addition of whole numbers and of rational numbers are related by the distributive property.

We have seen the distributive property in the form:

$a \times (b + c) = (a \times b) + (a \times c)$

Now look at the following:

$(b + c) \times a = a \times (b + c)$ Commutative property of multiplication.

$= (a \times b) + (a \times c)$ Distributive property

$= (b \times a) + (c \times a)$ Commutative property of multiplication.

i.e. $(b + c) \times a = (b \times a) + (c \times a)$

We see that the distributive property can be used when the single multiplier is on the left or right of the parentheses.

Now let us look at some examples in which the distributive property is used.

Examples

(1) $6 \times 37 = 6 \times (30 + 7) = (6 \times 30) + (6 \times 7) = 180 + 42 = 222$

(2) $6 \times 3\frac{1}{4} = 6 \times (3 + \frac{1}{4}) = (6 \times 3) + (6 \times \frac{1}{4}) = 18 + \frac{6}{4} = 18 + \frac{3}{2} = 19\frac{1}{2}$

(3) $\frac{1}{3} \times 6\frac{3}{5} = \frac{1}{3} \times (6 + \frac{3}{5}) = (\frac{1}{3} \times 6) + (\frac{1}{3} \times \frac{3}{5}) = 2 + \frac{1}{5} = 2\frac{1}{5}$

(4) $2\frac{1}{4} \times 5 = (2 + \frac{1}{4}) \times 5 = (2 \times 5) + (\frac{1}{4} \times 5) = 10 + \frac{5}{4} = 10 + 1\frac{1}{4} = 11\frac{1}{4}$

1. The product $3\frac{1}{3} \times 6\frac{1}{2}$ can be written as
$$\begin{array}{r} 6\frac{1}{2} \\ \times\ 3\frac{1}{3} \\ \hline \end{array}$$

Examine the following to see what has been done to find the product.

$$\begin{array}{r} 6\frac{1}{4} \\ \times\ 3\frac{1}{3} \\ \hline 2\frac{1}{12} \\ 18\frac{3}{4} \\ \hline 20\frac{10}{12} \end{array} = 20\frac{5}{6}$$

2. Use the same method as above to show that

$$4\frac{1}{3} \times 2\frac{1}{2} = 10\frac{5}{6}.$$

2A15

In the examples in items 1 and 2 on the previous page, we have also been using the distrubutive property. The following shows how.

$$3\frac{1}{3} \times 6\frac{1}{4} = (3 + \frac{1}{3}) \times 6\frac{1}{4} = (3 \times 6\frac{1}{4}) + (\frac{1}{3} \times 6\frac{1}{4}) = 3 \times (6 + \frac{1}{4}) + \frac{1}{3} \times (6 + \frac{1}{4})$$
$$= (3 \times 6) + (3 \times \frac{1}{4}) + (\frac{1}{3} \times 6) + (\frac{1}{3} \times \frac{1}{4}) = 18 + \frac{3}{4} + 2 + \frac{1}{12}$$
$$= 20 + \frac{9}{12} + \frac{1}{12} = 20 + \frac{10}{12} = 20\frac{5}{6}$$

Now do the exercises on the following activity page.

Answers: 1. $\frac{1}{3} \times 6\frac{1}{4} = \frac{1}{3} \times (6 + \frac{1}{4}) = (\frac{1}{3} \times 6) + (\frac{1}{3} \times \frac{1}{4}) = 2 + \frac{1}{12} = 2\frac{1}{12}$

$3 \times 6\frac{1}{4} = 3 \times (6 + \frac{1}{4}) = (3 \times 6) + (3 \times \frac{1}{4}) = 18 + \frac{3}{4} = 18\frac{3}{4}$

$2\frac{1}{12} + 18\frac{3}{4} = 20 + \frac{1}{12} + \frac{9}{12} = 20 + \frac{10}{12} = 20\frac{5}{6}$

2.
$$\begin{array}{r} 4\frac{1}{3} \\ \times 2\frac{1}{2} \\ \hline 2\frac{1}{6} \\ 8\frac{2}{3} \\ \hline 10\frac{5}{6} \end{array}$$

2A16

SECTION 6 - ACTIVITY

1. Use the distributive property to simplify the following:

(a) $(\frac{3}{4} \times \frac{2}{5}) + (\frac{3}{4} \times \frac{3}{5})$

(b) $(\frac{5}{8} \times \frac{5}{6}) + (\frac{5}{8} \times \frac{11}{6})$

(c) $(\frac{2}{3} \times \frac{3}{8}) + (\frac{2}{3} \times 2\frac{5}{8})$

(d) $(1\frac{3}{5} \times \frac{3}{4}) + (1\frac{3}{5} \times \frac{1}{4})$

(e) $(2\frac{1}{3} \times 1\frac{3}{5}) + (2\frac{1}{3} \times 1\frac{2}{5})$

(f) $(\frac{7}{8} \times \frac{1}{3}) + (\frac{1}{8} \times \frac{1}{3})$

(g) $(\frac{3}{10} \times \frac{1}{2}) + (\frac{17}{10} \times \frac{1}{2})$

(h) $(4\frac{1}{4} \times \frac{3}{8}) + (3\frac{3}{4} \times \frac{3}{8})$

(i) $(1\frac{2}{5} \times 1\frac{1}{3}) + (1\frac{3}{5} \times 1\frac{1}{3})$

2. Use the distributive property to find the products.

(a) $5 \times 6\frac{1}{5}$

(b) $8 \times 2\frac{3}{4}$

(c) $\frac{3}{4} \times 12\frac{1}{3}$

(d) $\frac{5}{8} \times 16\frac{2}{5}$

(e) $2\frac{1}{3} \times 6$

(f) $3\frac{2}{5} \times 10$

(g) $6\frac{1}{4} \times \frac{1}{2}$

(h) $9\frac{3}{8} \times \frac{2}{3}$

3. Find the following products without changing mixed numerals to fractions.

(a) $\begin{array}{r} 4\frac{1}{2} \\ \times\ 1\frac{1}{4} \\ \hline \end{array}$

(b) $\begin{array}{r} 12\frac{1}{3} \\ \times\ 2\frac{3}{4} \\ \hline \end{array}$

(c) $\begin{array}{r} 4\frac{1}{9} \\ \times 6\frac{3}{4} \\ \hline \end{array}$

(d) $\begin{array}{r} 10\frac{1}{4} \\ \times\ 3\frac{4}{5} \\ \hline \end{array}$

(e) $\begin{array}{r} 8\frac{1}{5} \\ \times\ 2\frac{5}{8} \\ \hline \end{array}$

(f) $\begin{array}{r} 15\frac{1}{4} \\ \times\ 4\frac{3}{5} \\ \hline \end{array}$

(g) $\begin{array}{r} 6\frac{3}{5} \\ \times\ 2\frac{2}{3} \\ \hline \end{array}$

SOLUTIONS

SECTION 5 - ACTIVITY

1. Whole numbers are closed for addition (ad and bc are both whole numbers from the previous line.)
2. Whole numbers are closed for multiplication.
3. Definition of multiplication of rational numbers.
4. Whole numbers are closed for multiplication.
5. bd is a non-zero whole number.
6. Definition of a rational number.
7. Multiplication of whole numbers is commutative.
8. $\frac{c}{d} \times \frac{a}{b}$
9. Multiplication of rational numbers is commutative.
10. $\frac{ad + bc}{bd}$
11. Multiplication of whole numbers is commutative.
12. $\frac{cb + da}{db}$
13. Definition of addition of rational numbers.
14. $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$
15. Definition of multiplication of rational numbers.
16. $\frac{a(ce)}{b(df)}$
17. Definition of multiplication of rational numbers
18. $\frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f})$
19. Multiplication of rational numbers is associative.
20. Definition of addition of rational numbers.
21. $\frac{a + 0}{b}$
22. 0 is the identity for addition of whole numbers.
23. $\frac{a}{b} + 0 = \frac{a}{b}$
24. $\frac{cf + de}{df}$
25. Definition of multiplication of rational numbers.
26. Distributive property for whole numbers.
27. Definition of multiplication of rational numbers.
28. Definition of addition of rational numbers.
29. Multiplication of whole numbers is associative.
30. Multiplication of whole numbers is commutative.
31. $(\frac{a}{b} \times \frac{c}{d}) + (\frac{a}{b} \times \frac{e}{f})$
32. $\frac{a \times 0}{b \times 1}$
33. Multiplication property of 0 for whole numbers and 1 is the identity for multiplication of whole numbers.
34. The multiplication property of 0 holds for the rational numbers.

2A18

ANSWERS

SECTION 6 - ACTIVITY

1. (a) $\frac{3}{4} \times (\frac{2}{5} + \frac{3}{5}) = \frac{3}{4} \times 1 = \frac{3}{4}$
   (b) $\frac{5}{8} \times (\frac{5}{6} + \frac{11}{6}) = \frac{5}{8} \times 2 = \frac{5}{4}$
   (c) $\frac{2}{3} \times (\frac{3}{8} + 2\frac{5}{8}) = \frac{2}{3} \times 3 = 2$
   (d) $1\frac{3}{5} \times (\frac{3}{4} + \frac{1}{4}) = 1\frac{3}{5} \times 1 = 1\frac{3}{5}$
   (e) $2\frac{1}{3} \times (1\frac{3}{5} + 1\frac{2}{5}) = 2\frac{1}{3} \times 3 = 7$
   (f) $(\frac{7}{8} + \frac{1}{8}) \times \frac{1}{3} = 1 \times \frac{1}{3} = \frac{1}{3}$
   (g) $(\frac{3}{10} + \frac{17}{10}) \times \frac{1}{2} = 2 \times \frac{1}{2} = 1$
   (h) $(4\frac{1}{4} + 3\frac{3}{4}) \times \frac{3}{8} = 8 \times \frac{3}{8} = 3$
   (i) $(1\frac{2}{5} + 1\frac{3}{5}) \times 1\frac{1}{3} = 3 \times 1\frac{1}{3} = 4$

2. (a) $5 \times (6 + \frac{1}{5}) = (5 \times 6) + (5 \times \frac{1}{5}) = 30 + 1 = 31$
   (b) $8 \times (2 + \frac{3}{4}) = (8 \times 2) + (8 \times \frac{3}{4}) = 16 + 6 = 22$
   (c) $\frac{3}{4} \times (12 + \frac{1}{3}) = (\frac{3}{4} \times 12) + (\frac{3}{4} \times \frac{1}{3}) = 9 + \frac{1}{4} = 9\frac{1}{4}$
   (d) $\frac{5}{8} \times (16 + \frac{2}{5}) = (\frac{5}{8} \times 16) + (\frac{5}{8} \times \frac{2}{5}) = 10 + \frac{1}{4} = 10\frac{1}{4}$
   (e) $(2 + \frac{1}{3}) \times 6 = (2 \times 6) + (\frac{1}{3} \times 6) = 12 + 2 = 14$
   (f) $(3 + \frac{2}{5}) \times 10 = (3 \times 10) + (\frac{2}{5} \times 10) = 30 + 4 = 34$
   (g) $(6 + \frac{1}{4}) \times \frac{1}{2} = (6 \times \frac{1}{2}) + (\frac{1}{4} \times \frac{1}{2}) = 3 + \frac{1}{8} = 3\frac{1}{8}$
   (h) $(9 + \frac{3}{8}) \times \frac{2}{3} = (9 \times \frac{2}{3}) + (\frac{3}{8} \times \frac{2}{3}) = 6 + \frac{1}{4} = 6\frac{1}{4}$

3. (a)
$$\begin{array}{r} 4\frac{1}{2} \\ \times\ 1\frac{1}{4} \\ \hline 1\frac{1}{8} \\ 4\frac{1}{2} \\ \hline 5\frac{5}{8} \end{array}$$

(b)
$$\begin{array}{r} 12\frac{1}{3} \\ \times\ 2\frac{3}{4} \\ \hline 9\frac{1}{4} \\ 24\frac{2}{3} \\ \hline 33\frac{11}{12} \end{array}$$

(c)
$$\begin{array}{r} 4\frac{1}{9} \\ \times\ 6\frac{3}{4} \\ \hline 3\frac{1}{12} \\ 24\ \frac{2}{3} \\ \hline 27\frac{9}{12} \end{array} = 27\frac{3}{4}$$

(d)
$$\begin{array}{r} 10\frac{1}{4} \\ \times\ 3\frac{4}{5} \\ \hline 8\frac{1}{5} \\ 30\frac{3}{4} \\ \hline 38\frac{19}{20} \end{array}$$

(e)
$$\begin{array}{r} 8\frac{1}{5} \\ \times\ 2\frac{5}{8} \\ \hline 5\frac{1}{8} \\ 16\frac{2}{5} \\ \hline 21\frac{21}{40} \end{array}$$

(f)
$$\begin{array}{r} 15\frac{1}{4} \\ \times\ 4\frac{3}{5} \\ \hline 9\frac{3}{20} \\ 61 \\ \hline 70\frac{3}{20} \end{array}$$

(g)
$$\begin{array}{r} 6\frac{3}{5} \\ \times\ 2\frac{2}{3} \\ \hline 4\frac{2}{5} \\ 13\frac{1}{5} \\ \hline 17\frac{3}{5} \end{array}$$

CHALLENGERS TOPIC 2

Think back to Problem Solving in Topic 1. It was stated that there are no 'hard-and-fast' rules to follow in the solving of all problems. Each problem will be different and will require its own 'line-of-attack'.

Sometimes a mathematical pattern or relationship can be discovered if we first look at several numerical examples. Here is an example in which looking at examples helps.

The first of two rational numbers is greater than zero, and the product of the two rational numbers is greater than zero.

(I) If the product is less than the first number, what must be true about the second number?

(II) If the product is greater than the first number, what must be true about the second number?

Let's look at some numerical examples to determine what must be true about the second number in each case.

(I)

First Number	Second Number	Product (less than first number)	Finding the Second Number	Second Number
5	n	3	$5 \times n = 3$ $\frac{1}{5} \times 5 \times n = \frac{1}{5} \times 3$ $n = \frac{3}{5}$	$\frac{3}{5}$
7	p	4	$7 \times p = 4$ $\frac{1}{7} \times 7 \times p = \frac{1}{7} \times 4$ $p = \frac{4}{7}$	$\frac{4}{7}$
$\frac{2}{3}$	q	$\frac{1}{4}$	$\frac{2}{3} \times q = \frac{1}{4}$ $\frac{3}{2} \times \frac{2}{3} \times q = \frac{3}{2} \times \frac{1}{4}$ $q = \frac{3}{8}$	$\frac{3}{8}$

What do you notice about all the second numbers?

You can look at as many examples as you require to find a pattern. Here we notice that every time the second number is less than one. Thus, it seems that if the product is less than the first number, then the second number must be less than one.

Now you try some examples for part (II) and see what is true for the second number in each case.

Read the following example very carefully.

'If a given rational number (less than one) is divided by a rational number between 0 and 1, how does the quotient compare in size with the given number?'

Let's try some numerical examples.

Given Number	Number Between 0 and 1	Finding Quotient	Quotient
$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4} \div \frac{1}{2}$ $\frac{3}{4} \times \frac{2}{1}$ $\frac{3}{2}$	$1\frac{1}{2}$
$\frac{3}{5}$	$\frac{4}{15}$	$\frac{3}{5} \div \frac{4}{15}$ $\frac{3}{5} \times \frac{15}{4}$ $\frac{9}{4}$	$2\frac{1}{4}$
$\frac{4}{7}$	$\frac{8}{11}$	$\frac{4}{7} \div \frac{8}{11}$ $\frac{4}{7} \times \frac{11}{8}$ $\frac{11}{14}$	$\frac{11}{14}$

How does the quotient compare in size with the given number? In each case we can see the quotient is greater than the given number, when the given number (less than one) is divided by a rational number between 0 and 1.

Remember that all problems are different. But you may sometimes find it useful to use numerical examples when looking for mathematical relationships or patterns.

The most important thing to do when solving problems is to THINK! THINK! THINK!

Read the following instructions and then do the problems.

1. After trying a problem, check your answer with the one given at the end of the Problem Solving section. If it is incorrect, try the problem again to see if you can arrive at the right answer.
2. Do NOT spend more than 15 to 20 minutes of hard thinking on a single problem. After that time, leave it and try the next problem in the section you are working. You are not expected to do every problem. If you find that the problems in a section are very easy and do not require you to do much thinking, leave this section and start the next section. The problems having been divided into four sections with the easier problems first. For you to be doing real problem solving you should be challenged and be required to think about a problem before being able to arrive at an answer.
3. When you have tried all the problems in a section, return to those you missed. You may be surprised at your second try.
4. Work on some of these problems at home. You may have more time to think about them.
5. Have your teacher look at your "Problem Solving" attempts and achievements.

Challengers Section 1.

1. Suppose you saw the two number sentences below written on the chalkboard by two students. What correct conclusion could you reach about these products without doing any computations? Why?

$$\frac{24}{23} \times \frac{65}{66} = \frac{1560}{1518} \qquad \frac{65}{66} \times \frac{24}{23} = \frac{1580}{1518}$$

2. Joe saw the two sentences below written on the chalkboard. He said "If both sentences are true, then addition of rational numbers is NOT commutative. Was he correct? Explain.

$$\frac{6}{8} + \frac{6}{12} = \frac{30}{24} \qquad \frac{6}{12} + \frac{6}{8} = \frac{5}{4}$$

3. Simplify the following:

a) $\frac{7}{35} + \frac{3}{18} + \frac{8}{105}$

b) $\frac{5}{24} + \frac{7}{30} + \frac{9}{40}$

4. What rational number less than 2 can be added to $\frac{5}{8}$ so that each sum is greater than 1?

5. Show how the distributive property can be used to show that $\frac{1}{2} \times 6\frac{2}{3}$ is $3\frac{1}{3}$.

6. The universe or replacement set for 'n' in the following conditions of equality is the set of whole numbers. Find replacements for 'n' which make each condition true.

a) $\frac{n}{n} = \frac{n}{8}$ b) $\frac{n}{n} = \frac{16}{16}$ c) $\frac{4}{5} \times \frac{3}{n} = \frac{3}{10}$

Challengers Section 2.

7. A certain rational number is greater than zero. Its reciprocal is larger than the number itself. What must be true about the number?

8. What can be said about the value of reciprocals of numbers which are very close to one?

9. If a given rational number (greater than zero) is multiplied by a rational number between 0 and 1, how does the product compare in size with the given number?

10. If a given rational number (greater than zero) is divided by a rational number between 0 and 1, how does the quotient compare in size with the given number?

Challengers Section 3.

11. Express the sum of $\frac{x}{y} + \frac{p}{q}$ as a fraction.

12. If a, b, and c are different whole numbers, find the values of a, b and c if:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

13. Write the solutions for the following sentences. The universe is the set of rational numbers greater than zero.

a) $\frac{3}{4} - n = \frac{1}{8}$

b) $n + \frac{2}{3} > \frac{5}{6}$

c) $n - \frac{2}{5} > \frac{1}{5}$

d) $2n > 3$

e) $\frac{2}{3} n > \frac{1}{6}$

14. Find the following sums. Then find a relationship between each sum and the numbers involved.

a) $\frac{1}{1x2} + \frac{1}{2x3} =$

b) $\frac{1}{1x2} + \frac{1}{2x3} + \frac{1}{3x4} =$

c) $\frac{1}{1x2} + \frac{1}{2x3} + \frac{1}{3x4} + \frac{1}{4x5} =$

d) $\frac{1}{1x2} + \frac{1}{2x3} + \frac{1}{3x4} + \frac{1}{4x5} + \frac{1}{5x6} =$

(cont.)

14. Use this relationship to find the following sums.

e) $\frac{1}{1\text{x}2} + \frac{1}{2\text{x}3} + \frac{1}{3\text{x}4} + \ldots + \frac{1}{49\text{x}50} =$

f) $\frac{1}{1\text{x}2} + \frac{1}{2\text{x}3} + \frac{1}{3\text{x}4} + \ldots + \frac{1}{99\text{x}100} =$

15. Find the sums of the following fractions and look for a relationship between the sum and the fractions.

eg. $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

a) $\frac{1}{9} + \frac{1}{18} =$ c) $\frac{1}{15} + \frac{1}{30} =$

b) $\frac{1}{12} + \frac{1}{24} =$ d) $\frac{1}{18} + \frac{1}{36} =$

Using the relationship found above, find the following sums. (Do NOT form the L.C.D. and add the fractions, use the relationship.)

e) $\frac{1}{21} + \frac{1}{42} =$ h) $\frac{1}{45} + \frac{1}{90}$

f) $\frac{1}{24} + \frac{1}{48} =$ i) $\frac{1}{48} + \frac{1}{96}$

g) $\frac{1}{30} + \frac{1}{60} =$

Complete the following sentences using the same pattern as above.

j) $\frac{1}{33} + \frac{1}{\square} = \frac{1}{\square}$

k) $\frac{1}{60} + \frac{1}{\square} = \frac{1}{\square}$

l) $\frac{1}{75} + \frac{1}{\square} = \frac{1}{\square}$

m) Write a formula to express the relationship found.

Challengers Section 4.

16. After much experimenting Robert claimed that dividing rational numbers could be done by dividing the numerators and dividing the denominators of the fractions in much the same way as multiplication of rational numbers. Try some numerical examples to determine if Robert was correct. Prove your answer using rational numbers $\frac{a}{b}$ and $\frac{c}{d}$.

17. Demonstrate on a number line the multiplication of the following rational numbers.

a) $\frac{3}{4} \times \frac{8}{9}$ b) $2 \times \frac{4}{3}$ c) $\frac{3}{2} \times \frac{3}{4}$

18. Indicate diagramatically, with regions, how to form the following products.

a) $\frac{2}{3} \times \frac{9}{4}$ b) $\frac{4}{3} \times \frac{3}{4}$ c) $\frac{4}{3} \times \frac{9}{4}$

19. a) Can the distributive property be used to find the product $6\frac{1}{2} \times 2\frac{2}{3}$? If it can, then show how.

b) If you were in the Advanced Group for this topic, prove the distributive property of multiplication over subtraction of rational numbers. (hint: use the definition of subtraction).

20.
$$\frac{11}{4} = 2 + \frac{3}{4} = 2 + \cfrac{1}{\frac{4}{3}} = 2 + \cfrac{1}{1\frac{1}{3}} = 2 + \cfrac{1}{1+\frac{1}{3}}$$

$2 + \cfrac{1}{1 + \frac{1}{3}}$ is known as a <u>continued</u> <u>fraction</u> for $\frac{11}{4}$.

It is represented by the symbols (2; 1, 3)

Another example of a continued fraction is:

$$\frac{19}{47} = 0 + \cfrac{1}{\frac{47}{19}} = 0 + \cfrac{1}{2\frac{9}{19}} = 0 + \cfrac{1}{2 + \frac{9}{19}} = 0 + \cfrac{1}{2 + \cfrac{1}{\frac{19}{9}}}$$

$$= 0 + \cfrac{1}{2 + \cfrac{1}{2\frac{1}{9}}} = 0 + \cfrac{1}{2 + \cfrac{1}{2+\frac{1}{9}}}$$

The continued fraction is represented by (0; 2, 2, 9).

* Each numerator in a continued fraction is <u>one</u>.

a) Compute the continued fraction for $\frac{64}{15}$ and write the symbol for it.

b) Which fraction is represented by the continued fraction (0; 1, 2, 3)?

Discussion:

If the product is greater than the first rational number, the second rational number must be greater than one.

Section 1

1. One of the products must be incorrect. According to the commutative property of multiplication for rational numbers, they should have equal products.
2. No, he was not correct.

$$\frac{30}{24} = \frac{5}{4} \quad \text{ie. } \frac{6}{8} + \frac{6}{12} = \frac{6}{12} + \frac{6}{8}$$

Thus the commutative property of addition for rational numbers is supported.

3. a) $\frac{31}{70}$ b) $\frac{2}{3}$
4. $\frac{3}{8} < n < 2$ ie. the set of rational numbers from $\frac{3}{8}$ to 2.
5. $\frac{1}{2} \times 6\frac{2}{3} = \frac{1}{2} \times (6 + \frac{2}{3}) = (\frac{1}{2} \times 6) + (\frac{1}{2} \times \frac{2}{3}) = 3 + \frac{1}{3} = 3\frac{1}{3}$.
6. a) 8 b) any whole number greater than zero c) 8

Section 2

7. The number is less than one.
8. The value of the reciprocals is also very close to one.
9. The product is smaller than the given number.
10. The quotient is greater than the given number.

Section 3

11. $\frac{xq + py}{yq}$ 12. $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$
13. a) sol: $\frac{5}{8}$ b) sol: all rational numbers greater than $\frac{1}{6}$.
    c) sol: all rational numbers greater than $\frac{3}{5}$.
    d) sol: all rational numbers greater than $\frac{3}{2}$.
    e) sol: all rational numbers greater than $\frac{1}{4}$.

14. a) $\frac{2}{3}$ b) $\frac{3}{4}$ c) $\frac{4}{5}$ d) $\frac{5}{6}$ e) $\frac{49}{50}$ f) $\frac{99}{100}$

15. a) $\frac{1}{6}$ b) $\frac{1}{8}$ c) $\frac{1}{10}$ d) $\frac{1}{12}$ e) $\frac{1}{14}$ f) $\frac{1}{16}$ g) $\frac{1}{20}$

h) $\frac{1}{30}$ i) $\frac{1}{32}$ j) $\frac{1}{33} + \frac{1}{66} = \frac{1}{22}$ k) $\frac{1}{60} + \frac{1}{120} = \frac{1}{40}$

l) $\frac{1}{75} + \frac{1}{150} = \frac{1}{50}$ m) $\frac{1}{3n} + \frac{1}{6n} = \frac{1}{2n}$

Section 4

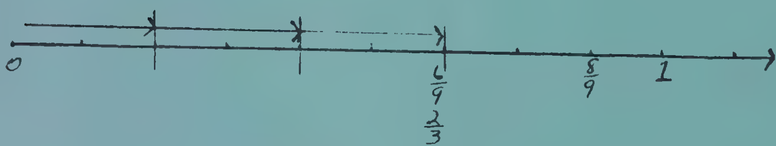
16. Robert was correct.

Usual method
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$$

Robert's method
$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d} = \frac{\frac{a}{c}}{\frac{b}{d}} = \frac{a}{c} \times \frac{d}{b} = \frac{a \times d}{c \times b} = \frac{a \times d}{b \times c}$$

Robert's method gives the same answer as the usual method.

17. a)

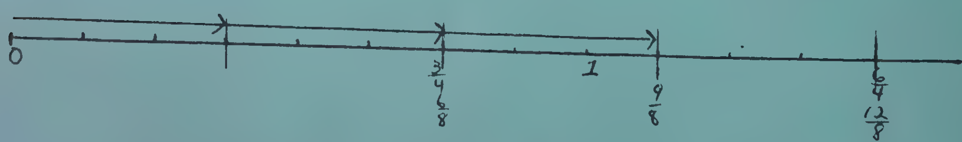


$\frac{3}{4} \times \frac{8}{9} = \frac{2}{3}$

b)

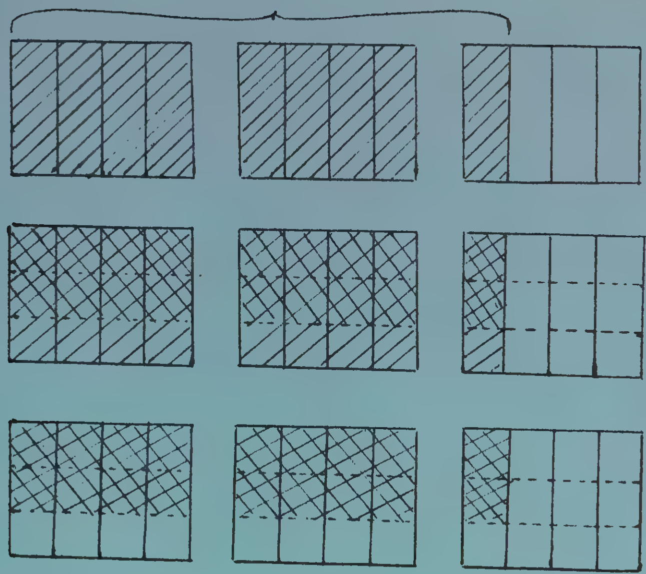
$2 \times \frac{4}{3} = \frac{8}{3}$

c)



$\frac{3}{2} \times \frac{3}{4} = \frac{9}{8}$

18. a)

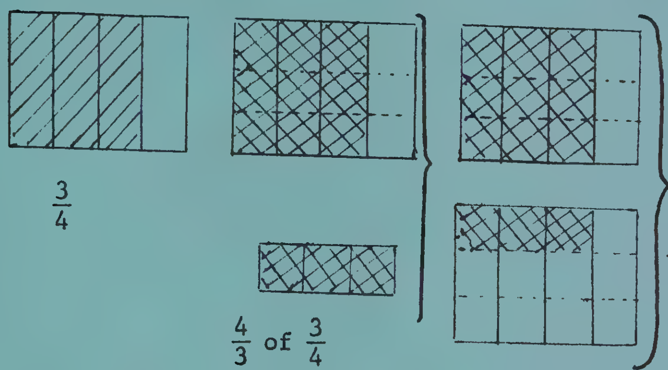


$\frac{9}{4}$

$\frac{2}{3}$ of $\frac{9}{4}$

$\frac{2}{3}$ of $\frac{9}{4} = \frac{18}{12}$

b)

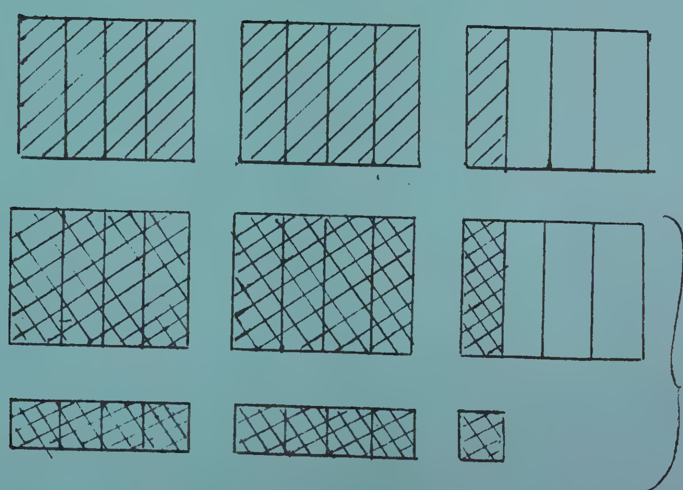


$\frac{3}{4}$

$\frac{4}{3}$ of $\frac{3}{4}$

$\frac{4}{3}$ of $\frac{3}{4} = \frac{12}{12}$

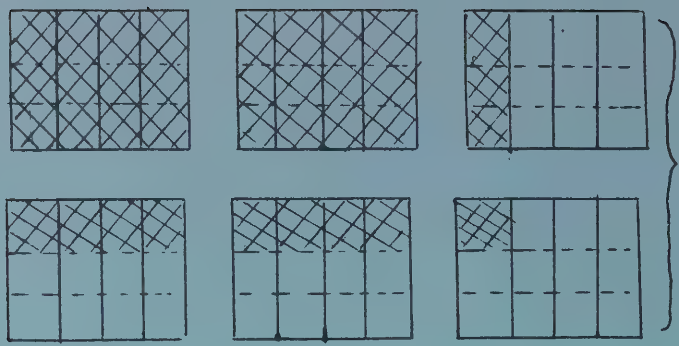
c)



$\frac{9}{4}$

$\frac{4}{3}$ of $\frac{9}{4}$

cont. c)



$\frac{4}{3}$ of $\frac{9}{4} = \frac{36}{12}$

19. a) $6\frac{1}{2} \times 2\frac{2}{3} = 6\frac{1}{2} \times (2 + \frac{2}{3})$

$$= (6\frac{1}{2} \times 2) + (6\frac{1}{2} \times \frac{2}{3})$$

$$= (6 + \frac{1}{2}) \quad \times \quad 2 \quad + (6 \times \frac{1}{2}) \times \frac{2}{3}$$

$$= (6 \times 2) + (\frac{1}{2} \times 2) + (6 \times \frac{2}{3}) + (\frac{1}{2} \times \frac{2}{3})$$

$$= \quad 12 \quad + \quad 1 \quad + \quad 4 \quad + \quad \frac{1}{3}$$

$$= \quad 17\frac{1}{3}$$

b) Let $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ be any rational numbers.

Prove: $\frac{a}{b} \times (\frac{c}{d} - \frac{e}{f}) = (\frac{a}{b} \times \frac{c}{d}) - (\frac{a}{b} \times \frac{e}{f})$

Proof:

Statement	Reason	Comments
$\frac{a}{b} \times (\frac{c}{d} - \frac{e}{f})$		Start with left side.
$= \frac{a}{b} \times (\frac{cf - de}{df})$	Definition of subtraction of rational numbers.	
$= \frac{a \times (cf - de)}{b \times (df)}$	Definition of multiplication of rational numbers.	
$= \frac{acf - ade}{bdf}$	Distributive property of whole numbers.	cf and de are whole numbers.
$(\frac{a}{b} \times \frac{c}{d}) - (\frac{a}{b} \times \frac{e}{f})$		Now start with right side and get it to be the same as the fourth line.
$= \frac{ac}{bd} - \frac{ae}{bf}$	Definition of multiplication of rational numbers.	
$= \frac{acbf - aebd}{bdbf}$	Definition of subtraction of rational numbers.	

Statement	Reason	Comments
$= \dfrac{b(acf - dae)}{bdbf}$	Distributive property of whole numbers.	
$= \dfrac{\cancel{b}(acf - dae)}{\cancel{b}dbf}$	Reducing the fraction.	
$= \dfrac{acf - ade}{bdf}$	Commutative property of whole numbers	Order is changed in some terms.
$\therefore \frac{a}{b} \times (\frac{c}{d} - \frac{e}{f}) = (\frac{a}{b} \times \frac{c}{d}) - (\frac{a}{b} \times \frac{e}{f})$		Result is same as left side.

ie. Multiplication is distributive over subtraction of rational numbers.

20. a) $$\frac{64}{15} = 4 + \frac{4}{15} = 4 + \cfrac{1}{\frac{15}{4}} = 4 + \cfrac{1}{3\frac{3}{4}} = 4 + \cfrac{1}{3 + \frac{3}{4}} = 4 + \cfrac{1}{3 + \cfrac{1}{\frac{4}{3}}} = 4 + \cfrac{1}{3 + \cfrac{1}{1\frac{1}{3}}} = 4 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{3}}}$$

$\therefore$ (4; 3, 1, 3)

b) $$(0;\ 1,\ 2,\ 3) = 0 + \cfrac{1}{1 + \cfrac{1}{2 + \frac{1}{3}}} = 0 + \cfrac{1}{1 + \cfrac{1}{2\frac{1}{3}}} = 0 + \cfrac{1}{1 + \cfrac{1}{\frac{7}{3}}} = 0 + \cfrac{1}{1 + \frac{3}{7}} = 0 + \cfrac{1}{1\frac{3}{7}} = 0 + \cfrac{1}{\frac{10}{7}} = 0 + \frac{7}{10}$$

$\therefore$ the fraction is $\frac{7}{10}$.

Topic II Operations With Rational Numbers

Post - Test I (Form A)

1. Show all your work in the spaces provided.
2. Place your answers or solutions on the lines (_______) whenever they are provided for this purpose on the right hand side of the page.
3. Your items will be marked wrong if your solution does not appear in the space indicated.
4. Work carefully and do not spend too much time on any one item. All items should be attempted.

I1.1 Find the sum for the following and write it as a basic fraction.

A. $\frac{2}{3} + \frac{7}{8} + \frac{3}{16}$

B.
$$\begin{array}{r} \frac{5}{9} \\ \frac{3}{4} \\ + \ \frac{7}{12} \\ \hline \end{array}$$

I1.1 A. (_______)

B. (_______)

I1.2 Find the difference for the following and write it as a basic fraction.

A. $\frac{7}{9} - \frac{3}{5} =$

B.
$$\begin{array}{r} \frac{17}{9} \\ - \ \frac{11}{12} \\ \hline \end{array}$$

I1.2 A. (_______)

B. (_______)

B2.1 A. Write each of the following as a mixed numeral:

i) $\frac{56}{9}$ B2.1 A. - i (______)

ii) $\frac{37}{5}$ ii (______)

B. Write each of the following mixed numerals as fractions.

i) $5\frac{5}{9}$ B2.1 B. - i (______)

ii) $6\frac{4}{7}$ ii (______)

I2.2 A. Show, using fractions, why the mixed numeral for $\frac{62}{7}$ is $8\frac{6}{7}$.

B. Use fractions to justify that the fraction for $7\frac{7}{9}$ is $\frac{70}{9}$.

B3.1 Find the sums and write each as a mixed numeral with the fraction part as a basic fraction.

A. $5\frac{3}{5}$

$+\ 6\frac{5}{8}$

B. $3\frac{1}{2} + \frac{1}{7} + 3\frac{1}{6}$

B3.1 - A.(______)

B.(______)

I3.2 Find the differences and write the fraction parts as basic fractions.

A. $2\frac{2}{3} - 1\frac{8}{11}$

B. $6\frac{5}{9}$

$-\ 3\frac{3}{7}$

I3.2 - A. (______)

B. (______)

I4.1 Solve the following condition and write the fraction in the solution as a basic fraction. Show the check.

$$5\frac{5}{6} = n + 3\frac{3}{10}$$

I4.1 (_______)

I5.1 John spends $\frac{2}{5}$ of an hr. on math. homework, then he spends $\frac{1}{6}$ of an hr. on English homework, next he spends $\frac{1}{12}$ of an hr. on French homework and finally he spends $\frac{3}{10}$ of an hr. on Social Studies. He intended to finish his homework within one hour. How much of one hour has he left?

I5.1 (_______)

I6.1 Using diagrams show how $\frac{3}{4}$ of $\frac{2}{3}$ can be obtained.

I7.1 Find the basic fraction for the following product.

$$3 \times \frac{34}{15} \times 1\frac{7}{17} \times \frac{25}{64}$$

I7.1 (_______)

I8.1 What relationship exists between two rational numbers which display the following property?

$$\frac{5}{8} \times \frac{8}{5} = 1?$$

I9.1 Find the following quotient:

$$\frac{\frac{16}{125}}{\frac{8}{25}}$$

I9.1 (______)

I9.2 Explain, using reciprocals, why $\frac{3}{8} \div 0$ is not possible.

I10.1 Solve the condition and show your check:

$$13\frac{1}{3}\ a = 23\frac{1}{3}$$

I10.1 (______)

I11.1 Susan got a large case of eggs at the store. On her way home she dropped the case. As a result $\frac{4}{5}$ of the case of eggs was damaged and of these $\frac{2}{3}$ were broken. The rest were cracked. What part of the case of eggs was cracked?

I11.1 (______)

I12.1 Give a numerical example for each which illustrates what it means to say that:

1) the set of rational numbers is closed under addition
2) the rational numbers have an identity element for addition
3) addition of rational numbers is associative
4) addition of rational numbers is commutative

1)
2)
3)
4)

I12.2 Give a numerical example for each which shows what it means to say that:

1) the set of rational numbers is closed under multiplication
2) every non-zero rational number has a reciprocal
3) multiplication of rational numbers is commutative
4) the rational numbers have the distributive property of multiplication over addition
5) multiplication of rational numbers is associative
6) the rational numbers have an identity element for multiplication

1)
2)
3)
4)
5)
6)

I12.3 Give an example to illustrate the property of multiplication the rational numbers have that the whole numbers do not have.

Topic II

Post-Test II - Basic (Form A)

1. Show all your work in the spaces provided.
2. Place your answers or solutions on the lines (_______) whenever they are provided for this purpose on the right hand side of the page.
3. Your items will be marked wrong if your solution does not appear in the space indicated.
4. Work carefully and do not spend too much time on any one item. All items should be attempted.

B1.1 Find the sums of the following and write each as a basic fraction.

A. $\frac{3}{5} + \frac{5}{6} + \frac{1}{3}$

B. $\begin{array}{r} \frac{3}{4} \\ \frac{1}{12} \\ + \frac{1}{6} \\ \hline \end{array}$

B1.1 A. (_______)

B. (_______)

B1.2 Find the differences of the following and write each as a basic fraction

A. $\frac{7}{5} - \frac{4}{3}$

B. $\begin{array}{r} \frac{16}{7} \\ - \frac{25}{14} \\ \hline \end{array}$

B1.2 A. (_______)

B. (_______)

B2.1 A. Write the fraction $\frac{39}{9}$ as a mixed numeral.

B. Write $9\frac{3}{4}$ as a fraction.

B2.1 A. (_______)

B. (_______)

B3.1 Find the sums of the following and write the fraction part in the answer as a basic fraction.

A. $11\frac{2}{3}$

$+12\frac{5}{8}$

B. $7\frac{7}{9} + 6\frac{2}{3} + 5\frac{5}{6}$

B3.1 A. (_______)

B3.1 B. (_______)

B3.2 Find the differences of the following and write the fraction part in the answer as a basic fraction.

A. $7\frac{7}{9} - 5\frac{5}{6}$

B. $18\frac{3}{7}$

$- 14\frac{2}{3}$

B3.2 A. (_______)

B3.2 B. (_______)

B4.1 Solve the condition $a - \frac{13}{14} = \frac{3}{28}$ and write the fraction in the solution as a basic fraction. Show your check.

B4.1 (_______)

B5.1 The batting average of the school team was $\frac{3}{10}$.
The batting average for Marvin, the short-stop, was $\frac{2}{5}$.
How much does the batting average of Marvin exceed the batting average of his team?

B5.1 (_______)

B7.1 Find the basic fraction for each product.

A. $\frac{5}{7}$ of $5\frac{3}{5}$ B7.1 A. (______)

B. $\frac{3}{14} \times \frac{28}{15} \times 10$ B7.1 B. (______)

B8.1 Give the reciprocal for each of the following numbers if it exists:

$\frac{4}{7}$; $\frac{8}{3}$; 0 ; 7 ; 1 ; $\frac{0}{12}$

B9.1 Find the quotients and give them as basic fractions:

A. $\frac{28}{27} \div \frac{7}{18}$

B. $$\frac{\frac{3}{15}}{\frac{12}{25}}$$

B9.1 A. (______)

B9.1 B. (______)

B10.1 Solve the condition $\frac{3}{5} = \frac{27}{35}$ a and write the solution as a basic fraction. Show the check.

B10.1 (______)

B11.1 Ann was to make a dress for graduation. She decided to shop around and found 2 kinds of materials she liked. Crimpoline was priced at $2\frac{2}{3}$ yards for $\frac{24}{25}$ dollar. Velveteen was priced at $1\frac{3}{4}$ yards for $\frac{14}{25}$ dollar. Which material was the cheapest per yard?

B11.1 (______)

Topic II

Post-Test II - Intermediate (Form A)

1. Show all your work in the spaces provided.
2. Place your answers or solutions on the lines (_______) whenever they are provided for this purpose on the right hand side of the page.
3. Your items will be marked wrong if your solution does not appear in the space indicated.
4. Work carefully and do not spend too much time on any one item. All items should be attempted.

I1.1 Find the sums of the following and write each as a basic fraction.

A. $\frac{5}{7} + \frac{3}{8} + \frac{9}{14}$

B. $\begin{array}{r} \frac{7}{15} \\ \frac{5}{9} \\ + \frac{3}{5} \\ \hline \end{array}$

I1.1 A. (_______)

I1.1 B. (_______)

I1.2 Find the differences of the following and write each as a basic fraction.

A. $\frac{17}{12} - \frac{4}{5}$

B. $\begin{array}{r} \frac{24}{25} \\ - \frac{2}{3} \\ \hline \end{array}$

I1.2 A. (_______)

I1.2 B. (_______)

I2.2 A. Justify that the mixed numeral for $\frac{47}{9}$ is $5\frac{2}{9}$ by using fractions

B. Use fractions to justify that the fraction for $11\frac{3}{7}$ is $\frac{80}{7}$

I3.2 Find the differences and write the fraction part in each answer as a basic fraction.

A. $14\frac{17}{45} - 12\frac{7}{15}$

B. $\begin{array}{r} 7\frac{8}{27} \\ - 6\frac{2}{3} \\ \hline \end{array}$

I3.2 A. (_______)

B. (_______)

I4.1 Solve the condition $9\frac{9}{16} = 3\frac{19}{24} + m$. Write the fraction in the solution as a basic fraction. Show the check.

I4.1 (_______)

I5.1 Sylvia wants to make a dress for graduation. She is short of funds and as a result has to go to the remnant sales. At one sale she gets 1 1/4 yards of material, at another sale she gets 7/8 yards of the same material. How much more material does she need if the dress pattern calls for 3 1/2 yards of cloth?

I5.1 (_______)

I6.1 Using diagrams show how $\frac{3}{4}$ of $\frac{2}{5}$ can be obtained.

I7.1 Find the products for each of the following and write the fractions in the answers as basic fractions.

A. $\frac{16}{21}$ of $4\frac{1}{12}$

B. $\frac{14}{15} \times 35 \times 1\frac{17}{49}$

I7.1 A. (_______)

I7.1 B. (_______)

I8.1 In the following illustrate with an example:

A. Explain why zero has no reciprocal.

B. State the property which a number and its reciprocal have.

C. (1) Give the reciprocal for 1

(2) Write the reciprocal for a rational number

(3) Give the reciprocal for a whole number other than zero.

I9.1 Find the quotient of the following:

A. $8\frac{2}{9} \div 4\frac{1}{9}$

B. $\dfrac{9\frac{1}{8}}{18\frac{1}{4}}$

I9.1 A. (_______)

I9.1 B. (_______)

I9.2 Using reciprocals explain why division by 0 is not possible.

I10.1 Solve the condition $\frac{63}{32}$ n = $\frac{35}{16}$. Write the fraction part in the solution as a basic fraction. Show the check.

I10.1 (______)

I11.1 John wants to improve the performance of his motorbike. He has a choice of two gasoline additives. Additive A comes in 4 1/2 oz. cans and costs 3/4 dollar per can. Additive B comes in 5 1/4 oz. cans and sells for 7/10 dollar per can. Which brand should he buy if he wants the lowest cost per ounce? Show all work.

I11.1 (______)

Topic II

Intermediate - Supplement

Post-test II

I12.1 Match the example given in column I with the correct property from column II for the properties of addition of rational numbers.

________	$\frac{1}{2} + \frac{2}{3}$ R	a.	Associative property
________	$\frac{1}{2} + 0 = \frac{1}{2}$	b.	Commutative property
________	$\frac{2}{3} + \frac{3}{4} = \frac{3}{4} + \frac{2}{3}$	c.	Closure property
________	$(\frac{1}{2} + \frac{1}{3}) + \frac{1}{4} = \frac{1}{2} + (\frac{1}{3} + \frac{1}{4})$	d.	Identity property

I12.2 Match the example given in column I with the correct property from column II for the properties of multiplication of rational numbers.

________	$\frac{1}{2} \times \frac{2}{3} = \frac{2}{3} \times \frac{1}{2}$	a.	Closure property
________	$\frac{1}{2} (\frac{1}{4} + \frac{1}{5}) = \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{5}$	b.	Associative property
________	$\frac{2}{3} \times \frac{3}{2} = 1$	c.	Identity property
________	$1 \times \frac{3}{4} = \frac{3}{4}$	d.	Commutative property
________	$\frac{1}{2} \times \frac{3}{4}$ R	e.	Reciprocal property
________	$(\frac{1}{2} \times \frac{1}{3}) \times \frac{1}{4} = \frac{1}{2} \times (\frac{1}{3} \times \frac{1}{4})$	F.	Distributive property

I12.3 State the property of multiplication the rational numbers have that the whole numbers do not have. Give an example to illustrate your statement.

TOPIC II

Post-Test II - Advanced (Form A)

Show your work in the spaces provided for this purpose. Work carefully and do not spend too much time on any one question.

A1 Define a non-negative rational number.

A2 Write the definitions for addition and subtraction of rational numbers named by fractions.

A3 Write the definition for multiplication of rational numbers named by fractions.

A4 Write the definition for division of rational numbers named by fractions.

A5 Complete the following proof.

Prove that multiplication is commutative for rational numbers.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two rational numbers.

<u>Prove</u>: $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$

<u>Proof</u>:

Statement	Reason
$\frac{a}{b} \times \frac{c}{d}$	Definition for multiplication of rational numbers
= (1) ________________	
	(2) ________________
$= \frac{ca}{db}$	
$= \frac{c}{d} \times \frac{a}{b}$	(3) ________________
∴ (4) ________________	

i.e. Multiplication of rational numbers is commutative.

A6 A. Show how the distributive property may be used to simplify

$$\left(\frac{7}{15} \times \frac{7}{38}\right) + \left(\frac{4}{5} \times \frac{7}{38}\right)$$

B. Show how the distributive property may be used to find the product

$\frac{3}{8} \times 24\frac{16}{25}$

C. Find the product without changing the mixed numerals to fractions.

$$\begin{array}{r} 8\frac{2}{3} \\ \times\ 2\frac{3}{8} \\ \hline \end{array}$$

A P P E N D I X D

TESTS AND ANALYSES

Final Version of Attitude Scale
Mathematics Achievement Test - Form I
Mathematics Achievement Test - Form II
Test Grid
Item Analysis - Form I
Item Analysis - Form II

FINAL VERSION OF ATTITUDE SCALE

A MATHEMATICS STUDY

The best answer to each statement is your own first impression. There are no right or wrong answers. Think carefully, but do not spend too much time on any one question. Let your own personal experience guide you to choose the answer you feel about each statement.

Please mark a response for every statement.

1. I find most mathematics lessons:

    A) extremely interesting.
    B) quite interesting.
    C) interesting.
    D) not very interesting.
    E) not interesting at all.

2. A knowledge of mathematics for any job at all is:

    A) most important.
    B) very important.
    C) quite important.
    D) of small importance.
    E) not important.

3. If I did not have to take mathematics, I would like school:

    A) much less.
    B) a little less.
    C) same as now.
    D) a little better.
    E) much better.

4. Mathematics is:

    A) the most important subject.
    B) one of the more important subjects.
    C) just as important as any other subject.
    D) not as important as some of the other subjects.
    E) the least important subject.

5. I find problem solving:

    A) extremely interesting.
    B) quite interesting.
    C) interesting.
    D) not very interesting.
    E) not interesting at all.

6. When I have difficulty with a new topic in my mathematics course, I ask my teacher to clarify the section:

   A) very frequently.
   B) frequently.
   C) sometimes.
   D) hardly ever.
   E) never.

7. If books about mathematics were available, I would:

   A) read most of them.
   B) read some of them.
   C) look at the diagrams and pictures.
   D) page through some of them.
   E) never look at them.

8. If someone says mathematics classes are worthless and a waste of time, I would:

   A) strongly disagree.
   B) tend to disagree.
   C) not take a side.
   D) tend to agree.
   E) strongly agree.

9. When I do my homework, my mathematics is:

   A) always done first.
   B) often done first.
   C) usually done first.
   D) sometimes done first.
   E) never done first.

10. I find mathematical puzzles:

   A) extremely interesting.
   B) quite interesting.
   C) sometimes interesting
   D) not very interesting.
   E) not interesting at all.

11. I would be interested in taking other subjects that make use of:

   A) a great deal of mathematics.
   B) quite a bit of mathematics.
   C) some mathematics.
   D) a little mathematics.
   E) no mathematics.

12. If given the opportunity to join one of the following clubs, I would prefer a:

A) mathematics club.
B) science club (physics).
C) science club (chemistry).
D) science club (geology).
E) literary club.

13. If I could receive one of the following magazines for a year, I would pick:

A) a mathematics magazine for high school students.
B) a magazine combining science and mathematics for high school students.
C) a science magazine for high school students.
D) a geology magazine for high school students.
E) a literary magazine for high school students.

14. When I study my mathematics course, I most often:

A) make written summaries of the sections covered.
B) do additional problem solving.
C) do many drill questions.
D) memorize the formulas given in the text.
E) look over some work done previoisly.

15. If I listed my courses in order of preference, I would place mathematics:

A) first.
B) second.
C) third.
D) fourth.
E) fifth

16. Whenever mathematical problems are presented to us for solving, I get:

A) a great deal of satisfaction in working them out.
B) quite a bit of satisfaction in working them out.
C) some satisfaction in working them out.
D) very little satisfaction in working them out.
E) no satisfaction in working them out.

17. My mathematics course has made:

A) mathematics enjoyable for me.
B) mathematics a pleasant course.
C) me feel indifferent towards mathematics.
D) mathematics classes an uncomfortable experience for me.
E) me strongly dislike mathematics.

18. I feel my mathematics teacher:

    A) enjoys teaching mathematics.
    B) gets some pleasure in teaching mathematics.
    C) gets some satisfaction in teaching mathematics.
    D) neither likes or dislikes teaching mathematics.
    E) dislikes teaching mathematics.

19. When I do my mathematics homework, I am usually:

    A) extremely interested.
    B) interested.
    C) somewhat interested.
    D) not too interested.
    E) not interested at all.

20. When we start a new topic in mathematics, I am usually:

    A) keenly interested.
    B) interested.
    C) somewhat interested.
    D) not too interested.
    E) not interested at all.

21. The average amount of time I spend on homework assignment in mathematics takes the following time per day:

    A) more than one hour.
    B) 3/4 hour to one hour.
    C) 1/2 hour to 3/4 hour.
    D) 1/4 hour to 1/2 hour.
    E) 0 hours to 1/4 hour.

22. When I get an assignment in mathematics:

    A) I do it immediately.
    B) I do it eventually.
    C) I may get it done.
    D) I put it off as long as possible.
    E) I don't do it.

23. Most of my work in this class is done:

    A) to satisfy my curiosity about mathematics.
    B) to gain competence in mathematics.
    C) to get a good mark.
    D) to just pass the class.
    E) to put in the time allotted to mathematics.

24. During mathematics lessons, I feel:

    A) extremely confident in myself.
    B) quite confident in myself.
    C) confident in myself.
    D) a little unsure of myself.
    E) very unsure of myself.

LT/mr

MATHEMATICS ACHIEVEMENT TEST - FORM I

1. Which of the following sets is a set of equivalent fractions?

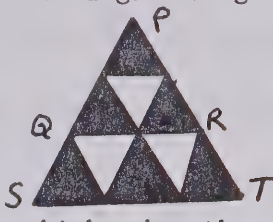
A. $\left\{\frac{1}{3}, \frac{3}{6}, \frac{6}{9}, \ldots\right\}$ C. $\left\{\frac{2}{3}, \frac{6}{9}, \frac{14}{21}, \ldots\right\}$

B. $\left\{\frac{3}{4}, \frac{9}{12}, \frac{16}{20}, \ldots\right\}$ D. $\left\{\frac{2}{5}, \frac{5}{10}, \frac{8}{15}, \ldots\right\}$

2. In the following diagram, the area of the small triangle PQR is what fraction of the shaded part of the large triangle PST?

A. $\frac{1}{2}$ C. $\frac{3}{2}$

B. $\frac{2}{3}$ D. $\frac{2}{1}$



3. In $\frac{2}{3} < \frac{n}{24} < \frac{3}{4}$, a whole number value for n which makes the statement true is

A. 15 C. 20

B. 17 D. 24

4. A rational number between the rational numbers $\frac{4}{3}$ and $\frac{5}{3}$ is

A. $\frac{5}{6}$ C. $\frac{9}{6}$

B. $\frac{7}{6}$ D. $\frac{11}{6}$

5. $\frac{2}{3} \times \frac{4}{5} \times \frac{10}{16} =$

A. $\frac{1}{3}$ C. $\frac{3}{2}$

B. $\frac{2}{3}$ D. $\frac{3}{1}$

6. $5 \div \frac{8}{10}$ is

A. 4 C. $\frac{4}{25}$

B. $6\frac{1}{4}$ D. $\frac{1}{4}$

7. $\frac{9}{4} + 1\frac{1}{4} - \frac{1}{2} =$

A. $3\frac{1}{4}$ C. $2\frac{1}{6}$

B. $2\frac{3}{4}$ D. 3

8. If any two numbers from the set $\left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ are called m and n; the number obtained by m*n will always represent a member of this set if * is replaced by

A. $-$ C. x

B. $+$ D. $\div$

9. The associative principle of multiplication of rational numbers is illustrated by

A. $(\frac{3}{4} \times \frac{4}{5}) \times \frac{5}{9} = \frac{5}{9} \times (\frac{4}{5} \times \frac{3}{4})$ C. $(\frac{3}{4} \times \frac{4}{5}) \times \frac{5}{9} = (\frac{4}{5} \times \frac{3}{4}) \times \frac{5}{9}$

B. $(\frac{3}{4} \times \frac{4}{5}) \times \frac{5}{9} = \frac{5}{9} \times (\frac{3}{4} \times \frac{4}{5})$ D. $(\frac{3}{4} \times \frac{4}{5}) \times \frac{5}{9} = \frac{3}{4} \times (\frac{4}{5} \times \frac{5}{9})$

10. $\frac{931}{100}$ expressed as a decimal numeral is

A. .931 C. 93.1

B. 9.31 D. 931

11. Solve: 58.61 + 8.511 + 106 + 14.4 = n

A. n = 187.521 C. n = 146.22

B. n = 264.12 D. n = 14.622

12. 2.88 ÷ .024 =

A. .12 C. 12

B. 1.2 D. 120

13. The weight of objects on the moon is .16 times earth weight. A rocket weighs 15.6 tons on earth. Its moon weight rounded off the the nearest ton would be

A. 2 tons C. 2.4 tons

B. 2.5 tons D. 3 tons

14. The correct decimal notation for $\frac{13}{32}$ is

A. .246 C. .40625

B. .406 D. 2.461

15. A salesman completed $\frac{2}{5}$ of a trip of 720 miles by plane, $\frac{1}{3}$ by car and the remainder by train. How many miles did he go by train?

A. 96 C. 270

B. 192 D. 528

16. 

Ken lives $\frac{2}{5}$ as far from school as John does. John lives $\frac{4}{5}$ of a mile from school. If Harry lives midway between John and Ken, how far is Harry's house from the school.

A. $\frac{8}{25}$ mile C. $\frac{12}{25}$ mile

B. $\frac{10}{25}$ mile D. $\frac{14}{25}$ mile

17. Two lengths are in the ratio of 3 to 7. If the smaller length is 21 inches, then the larger is

A. 25 inches C. 63 inches

B. 49 inches D. 84 inches

18. The ratio of the number of inches in a foot to the number of inches in a yard is

A. 3:1 C. 1:36

B. 1:3 D. 36:1

19. What number is 16% of 425?

A. 68 C. 2656.25

B. 92 D. 26.5625

20 After one year a car is worth 75% of its original cost. If the value of a car one year old is $2400, then the original cost was

A. $1800 C. $3000

B. $2720 D. $3200

21. Four students wrote a test in which each of the students was assigned a value. Which of the following students scored highest?

    A. Louise, who answered $\frac{4}{5}$ of the questions correctly.

    B. Andrew, who answered 83% of the questions correctly.

    C. Jane, who received 60 marks out of a possible eighty.

    D. Dean, who completed .81 of the questions correctly.

22. $\frac{1}{8}$ expressed as a percent is

    A. 2.5%   C. 8%

    B. 12.5%   D. 8.25%

23. Jack and Bob were practicing free shots. Jack got 17 out of 20. Bob got 27 out of 30. Judging from the evidence given, which of the statements tells us who is the better shooter?

    A. Jack missed 3 and Bob missed 3; thus they did equally well.

    B. Jack made 10 less than Bob; thus Bob did better.

    C. Jack made 17 and Bob made 27; thus Bob did better.

    D. Jack made 85% of his shots, while Bob made 90% of his shots; thus Bob did better.

24. The price of a motorcycle that had been used as a demonstrator was reduced by 20% of the list price. The motorcycle still did not sell, and the price was further reduced by 10% of the sale price. If the motorcycle was sold finally for \$720, what was the original list price?

    A. \$800   C. \$900

    B. \$1000   D. \$1100

MATHEMATICS ACHIEVEMENT TEST - FORM II

1. Which of the following sets is a set of equivalent fractions?

   a. $\left\{\frac{2}{3}, \frac{6}{9}, \frac{9}{12}, \ldots\right\}$

   b. $\left\{\frac{1}{6}, \frac{3}{12}, \frac{6}{18}, \ldots\right\}$

   c. $\left\{\frac{2}{5}, \frac{8}{20}, \frac{10}{25}, \ldots\right\}$

   d. $\left\{\frac{3}{4}, \frac{9}{16}, \frac{15}{24}, \ldots\right\}$

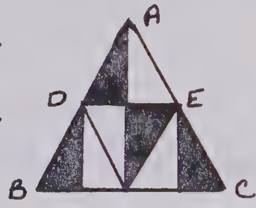
2. In the following diagram, the area of the small triangle ADE is what fraction of the shaded area of the large triangle ABC?

   a. $\frac{2}{3}$

   b. $\frac{1}{4}$

   c. $\frac{1}{3}$

   d. $\frac{1}{2}$



3. In $\frac{3}{4} < \frac{n}{40} < \frac{4}{5}$, a whole number value for n which makes the statement true is

   a. 24

   b. 27

   c. 31

   d. 36

4. A rational number between the rational numbers $\frac{7}{5}$ and $\frac{8}{5}$ is:

   a. $\frac{3}{2}$

   b. $\frac{5}{3}$

   c. $\frac{5}{4}$

   d. $\frac{7}{4}$

5. $\frac{3}{4} \times \frac{6}{7} \times \frac{14}{15} =$

   a. $\frac{1}{5}$

   b. $\frac{3}{5}$

   c. $\frac{4}{5}$

   d. $\frac{5}{3}$

6. $6 \div \frac{9}{12} =$

   a. $\frac{1}{8}$

   b. $\frac{2}{9}$

   c. $4\frac{1}{2}$

   d. 8

7. $\frac{7}{4} + 1\frac{3}{4} - \frac{1}{2} =$

a. $3\frac{1}{4}$ c. $2\frac{1}{6}$

b. $2\frac{3}{4}$ d. 3

8. If any two numbers from the set $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right\}$ are called x and y; the number obtained by x*y will always represent a number of this set if * is replaced by:

a. x c. +

b. ÷ d. -

9. The commutative property of addition of rational numbers is illustrated by:

a. $\frac{2}{3} + (\frac{4}{5} + \frac{5}{6}) = (\frac{2}{3} + \frac{4}{5}) + \frac{5}{6}$ c. $\frac{2}{3} + (\frac{4}{5} + \frac{5}{6}) = (\frac{2}{3} + \frac{4}{5}) + (\frac{2}{3} + \frac{5}{6})$

b. $\frac{2}{3} + (\frac{4}{5} + \frac{5}{6}) = \frac{2}{3} + \frac{4}{5} + \frac{5}{6}$ d. $\frac{2}{3} + (\frac{4}{5} + \frac{5}{6}) = \frac{2}{3} + (\frac{5}{6} + \frac{4}{5})$

10. $\frac{851}{100}$ expressed as a decimal numeral is

a. .851 c. 85.1

b. 8.51 d. 851

11. Solve: 14.27 + 8.352 + 109 - 5.4 = n

a. n = 131.622 c. n = 137.022

b. n = 126.222 d. n = 103.29

12. 5.46 ÷ .021 =

a. .26 c. 26

b. 2.6 d. 260

13. Floor covering costs $.37 per square foot. Find, to the nearest dollar, the cost of 45.2 square feet of this floor covering.

a. $16 c. $15.73

b. $16.72 d. $17

14. The correct decimal notation for $\frac{23}{32}$ is:

a. 1.3913 c. .718

b. .71875 d. 1.391

15. Jim travelled $\frac{2}{3}$ of a 480 mile trip by car, $\frac{1}{5}$ by plane and the rest by bus. How far did he travel by bus?

a. 64 miles c. 128 miles

b. 96 miles d. 160 miles

16. Town C is $\frac{4}{5}$ of the way to Town D from Town A. Town B is half way between Town A and Town C. It is 8 miles from Town C to Town D. How far is Town B from Town A?

a. 10 miles c. 20 miles

b. 16 miles d. $3\frac{1}{5}$ miles

17. Two lengths are in the ratio of 4 to 9. If the smaller length is 16 inches, then the larger is:

a. 36 inches c. 32 inches

b. 27 inches d. 54 inches

18. The ratio of the number of inches in a yard to the number of inches in a foot is:

a. 36:1 c. 3:1

b. 1:36 d. 1:3

19. What number is 13% of 195?

a. 15 c. 25.35

b. 1500 d. 2535

20. A farmer has 60% of his wheat left to sell at the end of the year. How much wheat did he have, if he has 2400 bushels left?

a. 1440 bu. c. 4000 bu.

b. 3000 bu. d. 3840 bu.

21. Four students were nominated for class president. Determine from the following statements, which student received the most votes and thus became class president.

    a. Joan, who received $\frac{6}{25}$ of the votes.

    b. Wayne, who received .25 of the votes.

    c. Bill, who received 23% of the votes.

    d. Jean, who received 12 out of 40 votes.

22. $\frac{7}{8}$ expressed as a per cent is

    a. 87.5% c. 78.5%

    b. 82.25% d. 85.75%

23. John and Tim were throwing darts. John missed the 'bull's-eye' 3 times out of 12, and Tim missed it 4 times out of 16. Judging from the evidence given, which of the following statements tells us who did best?

    a. John hit the 'bull's-eye' 9 times, and Tim hit it 12 times; thus Tim did better.

    b. John missed 1 less time than Tim; thus John did better.

    c. John made 75% of his throws and Tim made 75% of his throws; thus they did equally well.

    d. John made 3 less throws than Tim did; thus Tim did better.

24. At the end of the year, the cost of a particular model car increased 20% due to increased production costs. The next year it went up a further 10% and sold for $3300. What was the cost of the car before the increases?

    a. $2500 c. $2300

    b. $3000 d. $3100

TEST GRID

MATHEMATICS ACHIEVEMENT FORM I

Obj.	Know-ledge	Compre-hension	Appli-cation	Analysis Synthesis	Total
1		#1		#2	2
2		#3 #4			2
3		#5 #6 #7			3
4	#9			#8	2
5	#10				1
6		#11 #12	#13		3
7		#14			1
8			#15	#16	2
9		#17 #18			2
10		#19	#20		2
11		#22	#21		2
12			#23	#24	2
Total	2	13	5	4	24

Knowledge - 2/24 = 8.33%

Comprehension - 13/24 = 54.16%

Application - 5/24 = 20.83%

Analysis & Synthesis - 4/24 = 16.67%

ITEM ANALYSIS - FORM I

Exp. Group Item	N	Dif.	Bis Corr.	Item Rel.
1	257	.203	.489	.197
2		.496	.284	.142
3		.285	.486	.219
4		.508	.468	.234
5		.602	.412	.212
6		.262	.355	.156
7		.414	.472	.232
8		.297	.434	.198
9		.242	.393	.168
10		.285	.476	.215
11		.398	.563	.276
12		.137	.620	.213
13		.211	.288	.117
14		.098	.419	.124
15		.262	.408	.180
16		.176	.421	.160
17		.463	.364	.182
18		.331	.545	.256
19		.310	.322	.149
20		.416	.443	.218
21		.431	.469	.232
22		.225	.609	.255
23		.361	.516	.248
24		.264	.272	.120
Control Group				
1	203	.271	.478	.212
2		.453	.094	.047
3		.330	.520	.245
4		.557	.481	.239
5		.482	.386	-.192
6		.352	.542	.254
7		.483	.388	.194
8		.443	.218	.108
9		.172	-.022	-.008
10		.369	.461	.222
11		-.512	-.433	.216
12		.158	.380	.139
13		.207	.223	.090
14		.148	.358	.127
15		.281	.245	.110
16		.276	.538	.241
17		.550	.461	.229
18		.515	.597	.298
19		.356	.481	.230
20		.448	.440	.219
21		.493	.572	.289
22		.372	.595	.288
23		.303	.723	.332
24		.323	.116	.054

ITEM ANALYSIS - FORM II

Item	N	Dif.	Bis Corr.	Item Rel.
Exp. Group				
1	258	.447	.626	.311
2		.307	-.021	-.009
3		.436	.634	.314
4		.619	.568	.276
5		.681	.532	.248
6		.490	.624	.312
7		.728	.632	.281
8		.525	.271	.135
9		.272	.365	.162
10		.755	.555	.239
11		.607	.449	.219
12		.494	.527	.263
13		.416	.564	.278
14		.432	.665	.330
15		.479	.473	.237
16		.171	.196	.074
17		.634	.536	.258
18		.609	.634	.309
19		.449	.507	.252
20		.494	.391	.195
21		.538	.544	.271
22		.528	.673	.336
23		.641	.632	.303
24		.184	.261	.101
Control Group				
1	216	.444	.554	.275
2		.319	.133	.062
3		.440	.516	.256
4		.380	.656	.318
5		.486	.599	.299
6		.398	.581	.284
7		.653	.692	.330
8		.481	.440	.220
9		.319	.195	.091
10		.532	.531	.265
11		.537	.477	.238
12		.287	.558	.252
13		.315	.428	.199
14		.423	.488	.241
15		.460	.259	.129
16		.074	.228	.060
17		.631	.593	.286
18		.620	.552	.268
19		.493	.497	.249
20		.536	.463	.231
21		.457	.613	.306
22		.505	.640	.320
23		.619	.634	.308
24		.172	.209	.079